MATHEMATICAL ECONOMICS AND
ECONOMETRICS

VI SEMESTER

CORE COURSE

BA ECONOMICS

(2011 Admission)

UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

Calicut university P.O, Malappuram Kerala, India 673 635.
UNIVERSITY OF CALICUT

SCHOOL OF DISTANCE EDUCATION

STUDY MATERIAL

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BA ECONOMICS

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Prepared by: Module I & II Sri.Krishnan Kutty .V,
Assistant professor,
Department of Economics,
Government College, Malappuram.

Module III Sri.Sajeev. U,
Assistant professor,
Department of Economics,
Government College, Malappuram.

Module IV & V Dr. Bindu Balagopal,
Head of the Department,
Department of Economics,
Government Victoria College,
Palakkad.

Scrutinized by: Dr. C. Krishnan
Associate Professor,
PG Department of Economics,
Government College, Kodanchery,
Kozhikode – 673 580.

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Mathematical Economics and Econometrics

a. Introduction

Mathematical economics is an approach to economic analysis where mathematical symbols and theorems are used. Modern economics is analytical and mathematical in structure. Thus the language of mathematics has deeply influenced the whole body of the science of economics. Every student of economics must possess a good proficiency in the fundamental methods of mathematical economics. One of the significant developments in Economics is the increased application of quantitative methods and econometrics. A reasonable understanding of econometric principles is indispensable for further studies in economics.

b. Objectives

This course is aimed at introducing students to the most fundamental aspects of mathematical economics and econometrics. The objective is to develop skills in these. It also aims at developing critical thinking, and problem-solving, empirical research and model building capabilities.

c. Learning Outcome

The students will acquire mathematical skills which will help them to build and test models in economics and related fields. The course will also assist them in higher studies in economics.

d. Syllabus

Module I: Introduction to Mathematical Economics
Mathematical Economics: Meaning and Importance- Mathematical Representation of Economic Models- Economic functions: Demand function, Supply function, Utility function, Consumption function, Production function, Cost function, Revenue function, Profit function, Saving function, Investment function Marginal Concepts: Marginal utility, Marginal propensity to Consume, Marginal propensity to Save, Marginal product, Marginal Cost, Marginal Revenue, Marginal Rate of Substitution, Marginal Rate of Technical Substitution Relationship between Average Revenue and Marginal Revenue- Relationship between Average Cost and Marginal Cost - Elasticity: Demand elasticity, Supply elasticity, Price elasticity, Income elasticity, Cross elasticity- Engel function.

Module II: Constraint Optimization, Production Function and Linear Programming

Module III: Market Equilibrium
Market Equilibrium: Perfect Competition- Monopoly- Discriminating Monopoly

Module IV: Nature and Scope of Econometrics
Econometrics: Meaning, Scope, and Limitations - Methodology of econometrics - Types of data: Time series, Cross section and panel data.

Module V: The Linear Regression Model


Reference:
3. R.G.D Allen, Mathematical Economics
4. Mehta and Madnani -Mathematics for Economics
5. Joshi and Agarwal- Mathematics for Economics
6. Taro Yamane- Mathematics for Economics
8. Koutsoyiannis; Econometrics.
MODULE 1

INTRODUCTION TO MATHEMATICAL ECONOMICS

Mathematical Economics: Meaning and importance- Mathematical representation of Economic Models- Economic Function: Demand function, Supply function. Utility function, Consumption function, Production function, Cost function, Revenue function, Profit function, saving function, Investment function. Marginal Concepts: Marginal propensity to Consume, Marginal propensity to Save, Marginal product, Marginal Cost, Marginal revenue, Marginal Rate of Substitution, Marginal Rate of Technical Substitution. Relationship between Average revenue and Marginal revenue- Relationship between Average Cost and Marginal Cost- Elasticity: Demand elasticity, Supply elasticity, Price elasticity, Income elasticity Cross elasticity –Engel function.

1.1 Mathematical Economics

Mathematical Economics is not a distinct branch of economics in the sense that public finance or international trade is. Rather, it is an approach to Economic analysis, in which the Economist makes use of mathematical symbols in the statement of the problem and also drawn up on known mathematical theorem to aid in reasoning. Mathematical economics insofar as geometrical methods are frequently utilized to derive theoretical results. Mathematical economics is reserved to describe cases employing mathematical techniques beyond simple geometry, such as matrix algebra, differential and integral calculus, differential equations, difference equations etc.

It is argued that mathematics allows economist to form meaningful, testable propositions about wide-range and complex subjects which could less easily be expressed informally. Further, the language of mathematics allows Economists to make specific, positive claims about controversial subjects that would be impossible without mathematics. Much of Economics theory is currently presented in terms of mathematical Economic models, a set of stylized and simplified mathematical relationship asserted to clarify assumptions and implications.

1.2 The Nature of Mathematical Economics

As to the nature of mathematical economics, we should note that economics is unique among the social sciences to deal more or less exclusively with metric concepts. Price, supply and demand quantities, income, employment rates, interest rates, whatever studied in economics, are naturally quantitative metric concepts, where other social sciences need contrived concepts in order to apply any quantitative analysis. So, if one believes in systematic relations between metric concepts in economic theory, mathematical is a natural language in which to express them. However, mathematical as a language is a slightly deceptive parable, as Allen points out in his preface. If it were merely a language, such as English, a mathematical text should be possible to translate in to verbal English. Schumpeter too is keen to point out that mere representation of facts by figures, as in Francois Quesnay’s “Tableau Economique ‘or Karl Marx’s “reproduction schemes” is not enough for establishing a Mathematical Economics.
1.3 Mathematical Versus Nonmathematical Economics

Since Mathematical Economics is merely an approach to economic analysis, it should not and does not differ from the non mathematical approach to economic analysis in any fundamental way. The purpose of any theoretical analysis, regardless of the approach, is always to derive a set of conclusions or theorems from a given set of assumptions or postulates via a process of reasoning. The major difference between “mathematical economics” and “literary economics” lies principally in the fact that, in the former the assumptions and conclusions are stated in mathematical symbols rather than words and in equations rather than sentences; moreover, in place of literacy logic, use is made of mathematical theorems- of which there exists an abundance to draw upon – in the reasoning process.

The choice between literary logic and mathematical logic, again, is a matter of little import, but mathematics has the advantage of forcing analysts to make their assumptions explicit at every stage of reasoning. This is because mathematical theorems are usually stated in the “if then” form, so that in order to tap the ‘then” (result) part of the theorem for their use, they must first make sure that the “if” (condition) part does confirm to the explicit assumptions adopted. In short, that the mathematical approach has claim to the following advantages:

(a) The ‘language’ used is more concise and precise.

(b) There exists a wealth of mathematical theorems at our services.

(c) In forcing us to state explicitly all our assumptions as a prerequisite to the use of the mathematical theorems.

(d) It allows us to treating the general n-variable case.

1.4 Mathematical Economics versus Econometrics

The term Mathematical Economics is sometimes confused with a related term, Econometrics. As the ‘metric’ part of the latter term implies, Econometrics is concerned mainly with the measurement of economic data. Hence, it deals with the study of empirical observations using statistical methods of estimation and hypothesis testing.

Mathematical Economics, on the other hand, refers to the application of mathematical to the purely theoretical aspects of economic analysis, with a little or no concern about such statistical problems as the errors of measurement of the variable under study. Econometrics is an amalgam of economic theory, mathematical economics, economic statistics and mathematical statistics.

The main concern of Mathematical Economics is to express economic theory in mathematical form (equations) without regard to measurability or empirical verification of the theory. Econometrician is mainly interested in the empirical verification of economic theory. As we shall see, the Econometrician often uses the mathematical equations proposed by the mathematical economist but puts these equations in such a form that they lend themselves to empirical testing. And this conversion of mathematical in to econometric equations requires a great deal of ingenuity and practical skill.
1.5 Mathematical Representation of Economic Models

As economic model is merely a theoretical framework, and there is no inherent reason why it must be mathematical. If the model is mathematical, however, it will usually consist of a set of equations designed to describe the structure of the model. By relating a number of variables to one another in certain ways, these equations give mathematical form to the set of analytical assumptions adopted. Then, through application of the relevant mathematical operations to these equations, we may seek to derive a set of conclusions which logically follow from those assumptions.

1.5 Variable, Constant, and Parameters

A variable is something whose magnitude can change, i.e., something that can take on different values. Variables frequently used in economics include price, profit, revenue, cost, national income, consumption, investment, imports, and exports. Since each variable can assume various values, it must be represented by a symbol instead of a specific number. For example, we may represent price by $P$, profit by $\Pi$, revenue by $R$, cost by $C$, national income by $Y$, and so forth. When we write $P = 3$, or $C = 18$, however, we are “freezing” these variables at specific values.

Properly constructed, an economic model can be solved to give us the solution values of a certain set of variables, such as the market-clearing level of price, or the profit maximizing level of output. Such variables, whose solution values we seek from the model, are known as endogenous variables (originated from within). However, the model may also contain variables which are assumed to be determined by forces external to the model and whose magnitudes are accepted as given data only; such variables are called exogenous variables (originating from without). It should be noted that a variable that is endogenous to one model may very well be exogenous to another. In an analysis of the market determination of wheat price ($p$), for instance, the variable $P$ should definitely be endogenous; but in the framework of a theory of consumer expenditure, $p$ would become instead a datum to the individual consumer, and must therefore be considered exogenous.

Variables frequently appear in combination with fixed numbers or constants, such as in the expressions, $7P$ or $0.5R$.

A constant is a magnitude that does not change and is therefore the antithesis of a variable. When a constant is joined to a variable; it is often referred to as the coefficient of that variable. However, a coefficient may be symbolic rather than numerical.

As a matter of convention, parametric constants are normally represented by the symbols $a, b, c$, or their counterpart in the Greek alphabet $\alpha, \beta$ and $\lambda$. But other symbols naturally are also permissible.

1.6 Equation and Identities

Variables may exist independently, but they do not really become interesting until they are related to one another by equations or by inequalities. In economic equations, economist may distinguish between three types of equation: definitional equations, behavioral equations and conditional equations.
A **definitional equation** set up an identity between two alternate expressions that have exactly the same meaning. For such an equation, the identical- equality sign ≜ read; “is identically equal to” – is often employed in place of the regular equal sign =, although the latter is also acceptable. As an example, total profit is defined as the excess of total revenue over total cost; we can therefore write,

\[ \Pi = R - C \]

A **behavioral equation** specifies the manner in which a variable behaves in response to changes in other variables. This may involve either human behavior (such as the aggregate consumption pattern in relation to national income) or non human behavior (such as how total cost of a firm reacts to output changes.

Broadly defined, behavioral equations can be used to describe the general institutional setting of a model, including the technological (eg: production function) and legal (eg: tax structure aspects).

Consider the two cost function

\[ C = 75 + 10Q \quad \ldots \ldots \quad (1) \]
\[ C = 110 + Q^2 \quad \ldots \ldots \quad (2) \]

Since the two equations have different forms, the production condition assumed in each is obviously different from the others.

In equation (1), the fixed cost (the value of C when Q = 0) is 75, where as in (2), it is 110. The variation in cost is also different in (1), for each unit increases in Q, there are a constant increase of 10 in C, but in (2), as Q increase unit often unit, C will increase by progressively larger amounts.

A **Conditional equation** states a requirement to be satisfied, for example, in a model involving the notion of equilibrium, we must up an equilibrium condition, which describe the prerequisite for the attainment of equilibrium. Two of the most familiar equilibrium conditions in Economics is:

\[ Q_d = Q_s \]

(Quantity demanded equal to quantity supplied)

\[ S = I \]

(Intended saving equal to intended investment)

Which pertain respectively, to the equilibrium of a market model and the equilibrium of the national income model in its simplest form?
1.7 Economic Function

A function is a technical term used to symbolize relationship between variables. When two variables are so related, that for any arbitrarily assigned value to one of them, there correspond definite values (or a set of definite values) for the other, the second variable is said to be the function of the first.

1.8 Demand function

Demand function express the relationship between the price of the commodity (independent variable) and quantity of the commodity demanded (dependent variable). It indicate how much quantity of a commodity will be purchased at its different prices. Hence, \( d_x \) represent the quantity demanded of a commodity and \( p_x \) is the price of that commodity. Then,

Demand function \( d_x = f(p_x) \)

The basic determinants of demand function

\[ Q_x = f(P_x, P_r, Y, T, W, E) \]

Here,

- \( Q_x \): quantity demanded of a commodity \( X \)
- \( P_x \): price of commodity \( X \), \( P_r \): price of related good,
- \( Y \): consumer’s income,
- \( T \): Consumer/s tastes and preferences,
- \( W \): Consumer’s wealth,
- \( E \): Consumer’s expectations.

For example, the consumer’s ability and willingness to buy 4 ice creams at the price of Rs. 1 per ice-cream is an instance of quantity demanded. Whereas the ability and willingness of consumer to buy 4 ice creams at Rs. 1, 3 ice creams at Rs. 2 and 2 ice creams at Rs. 3 per ice-cream is an instance of demand.

Example: Given the following demand function

\[ Q_d = 720 - 25P \]

1.9 Supply function

Supply function express the relationship between the price of the commodity (independent variable) and quantity of the commodity supplied (dependent variable). It indicate how much quantity of a commodity that the seller offers at the different prices. Hence, \( S_x \) represent the quantity supplied of a commodity and \( p_x \) is the price of that commodity. Then,

Supply function \( S_x = f(p_x) \)
The basic determinants of supply function

\[ Q_s = f(G_f, P, I, T, P_r, E, G_p) \]

Here,
- \( Q_s \): quantity supplied,
- \( G_f \): Goal of the firm,
- \( P \): Product’s own price,
- \( I \): Prices of inputs,
- \( T \): Technology,
- \( P_r \): Prices of related goods,
- \( E \): Expectation of producer’s,
- \( G_p \): government policy.

**Example:** Given the following supply function

\[ Q_s = 720 - 25P \]

1.10 Utility function

People demand goods because they satisfy the wants of the people. The utility means wants satisfying power of a commodity. It is also defined as property of the commodity which satisfies the wants of the consumers. Utility is a subjective entity and resides in the minds of men. Being subjective it varies with different persons, that is, different persons derive different amounts of utility from a given good. Thus the utility function shows the relation between utility derived from the quantity of different commodity consumed. A utility function for a consumer consuming different goods may be represented:

\[ U = f(X_1, X_2, X_3, \ldots) \]

**Example:** For the utility function of two commodities

\[ U = f(x_1 - 2)^2(x_2 + 1)^3, \] find the marginal utility of \( x_1 \) and \( x_2. \)

\[ \frac{\partial U}{\partial x_1} = 2(x_1 - 2)(x_2 + 1)^3 \] is the MU function of the first commodity,

\[ \frac{\partial U}{\partial x_2} = 3(x_2 + 1)^2(x_1 - 2)^2 \] is the MU function of the second commodity

1.11 Consumption Function

The consumption function or propensity to consume denotes the relationship that exists between income and consumption. In other words, as income increases, consumers will spend part but not all of the increase, choosing instead to save some part of it. Therefore, the total increase in
income will be accounted for by the sum of the increase in consumption expenditure and the increase in personal saving. This law is known as propensity to consume or consumption function. Keynes contention is that consumption expenditure is a function of absolute current income, ie:

\[ C = f (Y_t) \]

The linear consumption function can be expressed as:

\[ C = C_0 + b Y_d \]

Where, \( C_0 \) is the autonomous consumption, \( b \) is the marginal propensity to consume and \( Y_d \) is the level of income.

Given the consumption function \( C = 40 + 0.80Y_d \), autonomous consumption function is 40 and marginal propensity to consume is \( 0.80 \).

1.12 Production function

Production function is a transformation of physical inputs into physical outputs. The output is thus a function of inputs. The functional relationship between physical inputs and physical output of a firm is known as production function. Algebraically, production function can be written as:

\[ Q = f (a,b,c,d, \ldots) \]

Where, \( Q \) stands for the quantity of output, \( a,b,c,d, \) etc; stands for the quantitative factors. This function shows that the quantity (q) of output produced depends upon the quantities, \( a, b, c, d \) of the factors \( A, B, C, D \) respectively.

The general mathematical form of Production function is:

\[ Q = f (L,K,R,S,v,e) \]

Where \( Q \) stands for the quantity of output, \( L \) is the labour, \( K \) is capital, \( R \) is raw material, \( S \) is the Land, \( v \) is the return to scale and \( e \) is efficiency parameters.

Example: Suppose the production function of a firm is given by:

\[ Q = 0.6X + 0.2Y \]

Where, \( Q \) = Output, \( X \) and \( Y \) are inputs.

1.13 Cost Function

Cost function derived functions. They are derived from production function, which describe the available efficient methods of production at any one time. Economic theory distinguishes between short run costs and long run costs. Short run costs are the costs over a period during which some factors of production (usually capital equipments and management) are fixed. The long run costs over a period long enough to permit the change of all factors of production. In
the long run all factors become variable. If $x$ is the quantity produced by a firm at a total cost $C$, we write for cost function as:

$$C = f(x)$$

It means that cost depends upon the quantity produced.

**Example:** The total cost function for producing a commodity in $x$ quantity is

$$TC = 60 - 12x + 2x^2$$

$$AC = \frac{TC}{x} = \frac{60 - 12x + 2x^2}{x} = \frac{60}{x} - 12 + 2x$$

$$MC = \frac{dTC}{dx} = -12 + 4x$$

### 1.14 Revenue Function

If $R$ is the total revenue of a firm, $X$ is the quantity demanded or sold and $P$ is the price per unit of output, we write the revenue function. Revenue function expresses revenue earned as a function of the price of good and quantity of goods sold. The revenue function is usually taken to be linear.

$$R = P \times X$$

Where $R = \text{revenue}$, $P = \text{price}$, $X = \text{quantity}$

If there are $n$ products and $P_1$, $P_2$, ..., $P_n$ are the prices and $X_1$, $X_2$, ..., $X_n$ units of these products are sold then

$$R = P_1X_1 + P_2X_2 + \ldots + P_nX_n$$

**Eg:**

$$TR = 100 - 5Q^2$$

**Example:** Given $P = Q^2 + 3Q + 1$, calculate $TR$

$$TR = P \times Q$$

$$P = (Q^2 + 3Q + 1) (Q)$$

$$P \times Q = Q^3 + 3Q^2 + Q$$
1.15 Profit Function

Profit function as the difference between the total revenue and the total cost. If x is the quantity produced by a firm, R is the total revenue and C being the total cost then profit (\(\pi\)).

\[
\Pi = TR - TC
\]

**Example:**

\[
TR = 300Q - 5Q^2
\]

\[
TC = 40Q^2 + 200Q + 200, \text{ find profit}
\]

Profit = TR – TC

\[
= 300Q - 5Q^2 - 40Q^2 - 200Q - 200
\]

\[
= 100Q + 9Q^2 - 200
\]

1.16 Saving Function

The saving function is defined as the part of disposable income which is not spending on consumption. The relationship between disposable income and saving is called the saving function. The saving function can be written as:

\[
S = f(Y)
\]

Where, S is the saving and Y is the income.

In mathematically the saving function is:

\[
S = c + bY
\]

Where S is the saving, c is the intercept and b is the slope of the saving function.

**Example:** Suppose a saving function is

\[
S = 30 + 0.4 \, Y
\]

1.17 Investment function

The investment function shows the functional relation between investment and the rate of interest or income. So, the investment function

\[
I = f (i)
\]

Where, I is the investment and i is the rate of interest

In other way, the investment function

\[
I = f ( Y_{t+1} - Y_t )
\]
Shows that I is the investment, $Y_{t+1}$ is future income, $Y_t$ is the present level of income. Investment is the dependent variable; the change in income is the independent variable.

### 1.18 Marginal Concepts

Marginal concept is concerned with variations of $y$ on the margin of $x$. That is, it is the variation in $y$ corresponding to a very small variation in $x$. ($x$ is the independent variable and $y$ depend upon it).

### 1.19 Marginal Utility

The concept of marginal utility was put forward by eminent economist Jevons. It is also called additional utility. The change that takes place in the total utility by the consumption of an additional unit of a commodity is called marginal utility. In other word, Marginal utility is the addition made to total utility by consuming one more unit of commodity. Supposing, by the consumption of first piece of bread you get 15 utile of utility and by the consumption of second piece of bread your total utility goes up to 25 utile. It means, the consumption of second piece of bread has added 10 utile (25 - 15) of utility to the total utility. Thus, the marginal utility of the second piece is 10 utile.

Marginal utility can be measured with the help of the following equation:

$$MU_{nth} = TU_n - TU_{n-1}$$

Or

$$MU = \frac{\Delta U}{\Delta q}$$

**Example:** Given the total utility function

$$U = 9xy + 5x + y$$

$$MU_x = \frac{\partial U}{\partial x} = 9y + 5$$

$$MU_y = \frac{\partial U}{\partial y} = 9x + 1$$

### 1.20 Marginal Propensity to Consume

Marginal propensity to consume measures the change in consumption due to a change in income of the consumer. In other words, MPC refers to the relationship between marginal income and marginal consumption. It may be the ratio of the change in consumption to the change in income. MPC is found by dividing the change in consumption by the change in income. MPC is the slope of the consumption line. Mathematically, MPC is the first derivative of the consumption function.

That is;

$$MPC = \frac{\Delta C}{\Delta Y}$$
Suppose, when income increases from Rs. 100 to Rs. 110, same time consumption increases from Rs. 75 to Rs. 80. Then the increment in income is Rs.10 and increment in consumption is Rs. 5. Thus,

\[ MPC = \frac{\Delta C}{\Delta Y} \]

\[ = \frac{5}{10} \]

\[ = .5 \]

**Example:** Suppose the consumption function is

\[ C = 24 + .8 Y \], find MPC

\[ MPC = \frac{\Delta C}{\Delta Y} \]

\[ = .8 \]

### 1.21 Marginal Propensity to Save

Marginal propensity to save is the amount by which saving changes in response to an incremental change in disposable income. In other words, Marginal propensity to save shows the how much of the additional income is devoted to saving. It measures the change in saving due to a change in income of the consumer. So the MPS is measure the ratio of change in saving due to change in income. MPS is the slope of the saving line. Mathematically MPS is the first derivative of the consumption function.

\[ MPS = \frac{\Delta S}{\Delta Y} \]

Suppose, when income increases from Rs. 150 to Rs. 200, same time saving increases from Rs. 50 to Rs. 80. Then the increment in income is Rs.50 and increment in saving is Rs. 80. Thus,

\[ MPC = \frac{\Delta S}{\Delta Y} \]

\[ = \frac{30}{50} \]

\[ = 0.6 \]

**Example:** Suppose the saving function is,

\[ S = 50 + .6 Y \], find MPS

\[ MPS = \frac{\Delta S}{\Delta Y} \]

\[ = .6 \]
1.22 Marginal Product

Marginal product of a factor of production refers to addition to total product due to the use of an additional unit of that factor.

1.23 Marginal Cost

Marginal cost is addition to the total cost caused by producing one more unit of output. In other words, marginal cost is the addition to the cost of producing \( n \) units instead of \( n-1 \) units, where \( n \) is any given number. In symbols:

\[
MC_n = TC_n - TC_{n-1}
\]

Suppose the production of 5 units of a product involve the total cost of Rs.206. If the increase in production to 6 units raises the total cost to Rs. 236, the marginal cost of the 6\(^{th} \) unit of output is Rs. 30. \((236 - 206 = 30)\)

Hence marginal cost is a change in total cost as a result of a unit change in output, it can also be written as:

\[
MC = \frac{\Delta TC}{\Delta Q}
\]

Where \( \Delta TC \) represents a change in total cost and \( \Delta Q \) represents a small change in output.

Example: Given the total cost function

\[TC = a + bQ + cQ^2 + dQ^3\]

Find the marginal cost.

\[
MC = \frac{dT C}{dQ} = b + 2cQ + 3dQ^2
\]

Example: \[TC = X^2 - 3 XY - Y^2\]

\[
MC = \frac{d(TC)}{dX} = 2X - 3Y
\]

\[
MC = \frac{d(TC)}{dY} = 3Y - 2Y
\]

1.24 Marginal Revenue

Marginal Revenue is the net revenue earned by selling an additional unit of the product. In other words, marginal revenue is the addition made to the total revenue by selling one more unit of
the good. Putting it in algebraic expression, marginal revenue is the addition made to total revenue by selling \( n \) units of a product instead of \( n-1 \) units where \( n \) is any given number.

If a producer sells 10 units of a product at price Rs.15 per unit, he will get Rs.150 as the total revenue. If he now increases his sales of the product by one unit and sells 11 units, suppose the price falls to Rs. 14 per unit. He will therefore obtain total revenue of Rs.154 from the sale if 11 units of the good. This means that 11th unit of output has added Rs.4 to the total revenue. Hence Rs. 4 is here the Marginal revenue.

\[
MR = \frac{\Delta TR}{\Delta Q}
\]

Where, \( \Delta TR \) stands for change in total revenue and \( \Delta Q \) stands for change in output.

**Example:** \( TR = 50Q - 4Q^2 \), Find MR

\[
MR = \frac{\Delta TR}{\Delta Q} = 50 - 8Q
\]

### 1.25 Marginal Rate of Substitution (MRS)

The Concept of marginal rate of substitution is an important tool of indifferent curve analysis of demand. The rate at which the consumer is prepared to exchange goods \( X \) and \( Y \) is known as marginal rate of substitution. Thus, the MRS of \( X \) for \( Y \) as the amount of \( Y \) whose loss can just be compensated by a unit gain in \( X \). In other words MRS of \( X \) for \( Y \) represents the amount of \( Y \) which the consumer has to give up for the gain of one additional unit of \( X \), so that his level of satisfaction remains the same.

Given the table, when the consumer moves from combination B to combination C, on his indifference schedule he forgoes 3 units of \( Y \) for the additional one unit gain in \( X \). Hence, the MRS of \( X \) for \( Y \) is 3.

Likewise, when the consumer moves from C to D, and then from D to E in indifference schedule, the MRS of \( X \) for \( Y \) is 2 and 1 respectively.

<table>
<thead>
<tr>
<th>Combination</th>
<th>Good X</th>
<th>Good Y</th>
<th>MRS x y</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>8</td>
<td>4:1</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>5</td>
<td>3:1</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>3</td>
<td>2:1</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
<td>2</td>
<td>1:1</td>
</tr>
</tbody>
</table>
In mathematical that MRS x y between goods is equal to the ratio of marginal utilities of good X and Y.

An indifference curve can be represented by:

\[ U (x, y) = a \]

Where, ‘a’ represent constant utility along an indifference curve. Taking total differential of the above, we have

\[
\frac{\partial u}{\partial x} \, dx + \frac{\partial u}{\partial y} \, dy = 0
\]

\[
- \frac{dy}{dx} = \frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}}
\]

\[ \frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial u}{\partial y} \] are the marginal utilities of goods X and Y respectively, thus

\[
- \frac{dy}{dx} = \frac{MU_x}{MU_y}
\]

- \( \frac{dy}{dx} \) is the negative slope of indifference curve and MRS x y. Thus

\[
MRS_{xy} = \frac{MU_x}{MU_y}
\]

Thus the MRS between two goods is equal to the ratio between the marginal utilities of two goods.

**Example:** Find \( MRS_{xy} \) for the function \( U = x^{3/4} y^{1/4} \)

\[
MU_x = \frac{\partial u}{\partial x}
\]

\[
= \frac{3}{4} x^{3/4-1} y^{1/4}
\]

\[
MU_y = \frac{\partial u}{\partial y}
\]

\[
= \frac{1}{4} x^{3/4} y^{1/4 - 1}
\]

\[
\frac{MU_x}{MU_y} = \frac{\frac{3}{4} x^{3/4-1} y^{1/4}}{\frac{1}{4} x^{3/4} y^{1/4 - 1}}
\]

\[
= 3 \times \frac{y}{x}
\]
Example: Find $MRS_{xy}$, $U = 8x + 4y$

$$MU_x = \frac{\partial U}{\partial x} = 8$$

$$MU_y = \frac{\partial U}{\partial y} = 4$$

$$MRS_{xy} = \frac{MU_x}{MU_y} = \frac{8}{4} = 2$$

1.26 Marginal Rate of Technical Substitution (MRTS)

Marginal rate of technical substitution indicates the ratio at which factors can be substituted at the margin without altering the level of output. It is the slope of the isoquant. More precisely, MRTS of labour for capital may be defined as the number of units of capital which can be replaced by one unit of labour the level of output remaining unchanged. The concept of MRTS can be easily understood from the table.

<table>
<thead>
<tr>
<th>Factor Combinations</th>
<th>Units of Labour</th>
<th>Units of Capital</th>
<th>MRTS of L for K</th>
</tr>
</thead>
<tbody>
<tr>
<td>A</td>
<td>1</td>
<td>12</td>
<td></td>
</tr>
<tr>
<td>B</td>
<td>2</td>
<td>8</td>
<td>4</td>
</tr>
<tr>
<td>C</td>
<td>3</td>
<td>5</td>
<td>3</td>
</tr>
<tr>
<td>D</td>
<td>4</td>
<td>3</td>
<td>2</td>
</tr>
<tr>
<td>E</td>
<td>5</td>
<td>2</td>
<td>1</td>
</tr>
</tbody>
</table>

Each of the inputs combinations A, B, C, D and E yield the same level of output. Moving down the table from combination A to combination B, 4 units of Capital are replaced by 1 unit of Labour in the production process without any change in the level of output. Therefore MRTS of labour for capital is 4 at this stage. Switching from input combination B to input combination C involves the replacement of 3 units of capital by an additional unit of labour, output remaining the same. Thus, the MRTS is now 3. Likewise, MRTS of labour for capital between factor combination D and E if 1.
The MRTS at a point on an isoquant (an equal product curve) can be known from the slope of the isoquant at the point. The slope of the isoquant at a point and hence the MRTS between factors drawn on the isoquant at that point.

An important point to the noted above the MRTS is that it is equal to the ratio of the marginal physical product of the two factors. Therefore, that MRTS is also equal to the negative slope of the isoquant.

\[
\text{MRTS} = \frac{\partial K}{\partial L} = \frac{\partial Q/\partial K}{\partial Q/\partial L} = \frac{\partial Q}{\partial K} \times \frac{\partial L}{\partial Q}
\]

Cancel \( \partial Q \) and \( \partial Q \)
\[
= \frac{\partial L}{\partial K} = \frac{\text{MPL}}{\text{MPK}}
\]

So the MRTS of labour for capital is the ratio of the marginal physical products of the two factors.

**Example:** The following production function is given below:
\[
Q = L^{0.75} K^{0.25}
\]
Find the MRTS \( L \ K \)

\[
\text{MRTS} = \frac{\text{MPL}}{\text{MPK}}
= \frac{\partial L}{\partial K}
\]

\( MP_L = \frac{\partial Q}{\partial L} \)
\[
= 0.75 L^{0.75 - 1} K^{0.25}
= 0.75 L^{-0.25} K^{0.25}
= 0.75 \left( \frac{K}{L} \right)^{0.25}
\]
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\[
MP_K = \frac{\partial Q}{\partial K} = 0.25L^{0.75}K^{-0.75-1} = 0.25L^{0.75}K^{-0.75} = 0.25\left(\frac{L}{K}\right)^{0.75}
\]

\[
MRTS = \frac{MP_L}{MP_K} = 0.75\left(\frac{L}{K}\right)^{0.25}/0.25\left(\frac{K}{L}\right)^{0.25} = 3\left(\frac{L}{K}\right)^{25}/\left(\frac{L}{K}\right)^{75} = 3K^{25}L^{-75}/L^{75}K^{-75} = 3K^{-25+75}/L^{75+25} = 3(K^1/L^1) = 3\frac{K}{L}
\]

1.27 Relationship between Average Revenue, Marginal Revenue

An important relationship between MR, AR (price) and price elasticity of demand which is extensively used in making price decisions by firms. This relationship can be proved algebraically also.

\[
MR = P (1 - \frac{1}{\varphi})
\]

Where \(P\) = Price and \(\varphi\) = point elasticity of demand.

MR is defined as the first derivative of total revenue (TR).

Thus, \(MR = \frac{\partial TR}{\partial Q} \) \hspace{1cm} (1)

Now, TR is the product price and \(Q\) is the quantity of the product sold

\(TR = P \times Q\),

Thus, \(MR = \frac{\partial P \cdot Q}{\partial Q} \)
Using the product rule of differentiation, of a product, we have

\[ MR = P \frac{dQ}{dQ} + Q \frac{dP}{dQ} \]

\[ = P + Q \frac{dP}{dQ} \] \hspace{1cm} (2)

This equation can be written as

\[ MR = P \left( 1 + \frac{Q}{P} \times \frac{dP}{dQ} \right) \] \hspace{1cm} (3)

Now, recall that point price elasticity of demand

\[ \frac{P}{Q} \times \frac{dQ}{dP} \]

It will thus be noticed that the expression

\[ \left( \frac{P}{Q} \times \frac{dQ}{dP} \right). \] Thus

\[ \frac{Q}{P} \times \frac{dP}{dQ} = \frac{1}{\varepsilon} \] \hspace{1cm} (4)

Substituting equation (4) in to equation (3), we obtain

\[ MR = P \left( 1 \pm \frac{1}{\varepsilon} \right) \]

Or

\[ = P \left( 1 - \frac{1}{\varepsilon} \right) \]

Price or P is the same thing as average revenue (AR)

Therefore:

\[ MR = AR \left( 1 - \frac{1}{\varepsilon} \right) \]

\[ = AR \left( \frac{\varepsilon - 1}{\varepsilon} \right) \]

Or

\[ AR = MR \left( \frac{\varepsilon}{\varepsilon - 1} \right) \]
1.28 Relationship between Average Cost and Marginal Cost

The relationship between the marginal cost and average cost is same as that between any other marginal—average quantities. When the marginal cost is less than average cost, average cost falls and when marginal cost is greater than average cost, average cost rises. This marginal—average relationship is a matter of mathematical truism and can be easily understood by a simple example. Suppose that a cricket player’s batting average is 50. If in his next innings he scores less than 50, say 45, then his average score will fall because his marginal (additional) score is less than his average score.

If instead of 45, he scores more than 50, say 55, in his next innings, then his average score will increase because now the marginal score is greater than his previous average score. Again, with his present average runs as 50, if he scores 50 also in his next innings than his average score will remains the same ie, 50, since his marginal score is just equal to the average score. Likewise, suppose a producer is producing a certain number of units of a product and his average cost is Rs. 20. Now, if he produces one unit more and his average cost falls, it means that the additional unit must have cost him less than Rs. 20. On the other hand, if the production of the additional unit raises his average production of an additional unit, the average cost remains the same, then marginal unit must have cost him exactly Rs. 20, that is, marginal cost and average cost would be equal in this case.

1.29 Elasticity

Elasticity of the function f(x) at the point x is defined as the rate of proportionate change in f(x) per unit proportionate change in x.

1.30 Demand elasticity

It is the price elasticity of demand which is usually referred to as elasticity of demand. But, besides price elasticity of demand, there are various concepts of demand elasticity. That the demand for a good will is determined by its price, income of the people, prices of related goods etc…. Quantity demanded of a good will change as a result of a change in the size of any of these determinants of demand. The concept of elasticity of demand therefore refers to the degree of responsiveness of quantity demanded of a good to a change in its price, income or prices of related goods. Accordingly, there are three kinds of demand elasticity: price elasticity. Income elasticity and cross elasticity.

1.31 Elasticity of Supply

The concept of elasticity of supply like the elasticity of demand occupies an important place in price theory. The elasticity of supply is the degree of responsiveness of supply to changes in the price of a good. More, precisely, the elasticity of supply can be defined as a relative change in quantity supplied of a good in response to a relative changes in price of the good. Therefore,

\[ Es = \frac{\Delta q/q}{\Delta p/p} \]
For an accurate measure of elasticity of supply at mid point method may be used

\[ E_s = \frac{q_2 - q_1}{q_1 + q_2} \div \frac{p_2 - p_1}{p_1 + p_2} \]

\[ q_2 - q_1 = q \]
\[ p_2 - p_1 = p \]

Therefore,

\[ E_s = \frac{q}{q_1 + q_2} \times \frac{p_1 + p_2}{p} \]

The elasticity of supply depends upon the ease with which the output of an industry can be expanded and the changes in marginal cost of production. Since there is greater scope for increase in output in the long run than in the short run, the supply of a good is more elastic in the long run than in the short run.

**Example:** Find the elasticity of supply when price 5 units. Supply function is given by 

\[ q = 25 - 4p + p^2 \]

\[ \frac{\partial q}{\partial p} = -4 + 2p \]

Price elasticity of supply = \(-p/q \times \frac{\partial p}{\partial q} \)

\[ = \frac{-p}{25 - 4p + p^2} \times (-4 + 2p) \]
\[ = \frac{4p - 2p^2/25 - 4p + p^2}{25 - 4p + p^2} \]

Elasticity when \( P = 5 \) is 

\[ 4 \times 5 - 2(5)^2/25 - 4 \times 5^2 \]
\[ = 20 - 50/25 - 20 +25 \]
\[ = -30/30 = -1 \]
\[ = -1 = 1 \text{ (numerically)} \]

Ie, Elasticity is unit at \( P = 5 \)
Example: When the price of refrigerator rises from Rs. 2000 per unit to Rs. 2500 per unit and in response to this rise is price the quantity supplied increases from 2500 units to 3500 units, find out the price elasticity of supply.

(Hint: Since the change in price is quite large, midpoint method should be used to measure elasticity of supply)

\[
\Delta q = 3500 - 2500 = 1000 \\
q_2 + q_1 / 2 = 3500 + 2500 / 2 \\
= 3000 \\
p = 2500 - 2000 \\
= 500 \\
p_1 + p_2 / 2 = 2500 + 2000 / 2 \\
= 2250 \\
Es: 1000/3000 \div 500/2250 \\
1000/3000 \times 2250/500 \\
= 1/3 \times 4.5 / 1 \\
= 4.5 / 3 \\
= 1.5
\]

1.32 Price Elasticity

Price elasticity of demand express the response of quantity demanded of a good to change in its price, given the consumer’s income, his tastes and prices of all other goods.

Price elasticity means the degree of responsiveness or sensitiveness of quantity demanded of a good to changes in its prices. In other words, price elasticity of demand is a measure of relative change in quantity purchased of a good in response to a relative change in its price.

\[
\text{Price elasticity} = \frac{\text{Proportionate change in quantity demanded}}{\text{Proportionate change in price}} \\
\text{Or} \\
= \frac{\text{Change in quantity demanded}}{\text{Change in price}}
\]

In symbolic term
\[ Ep = \frac{\Delta q}{q} \times \frac{p}{\Delta p} \]

Where,

Ep stands for price elasticity

q Stands for quantity

P stands for price

\( \Delta \) stands for change

Price elasticity of demand (Ep) is negative, since the change in quantity demanded is in opposite direction to the change in price. But for the sake of convenience in understanding to the change in price, we ignore the negative sign and take into account only the numerical value of the elasticity.

**Example:** Suppose the price of a commodity falls from Rs. 6 to Rs. 4 per unit and due to this quantity demanded of the commodity increases from 80 units to 120 units. Find the price elasticity of demand.

Solution: Change in quantity demand (\( Q_2 - Q_1 \))

\[ 120 - 80 = 40 \]

Percentage change in quantity demanded

\[ = \frac{Q_2 - Q_1}{Q_2 + Q_1/2} \times 100 \]

\[ = \frac{40}{200}/2 \times 100 \]

\[ = 40 \]

Change in price \( p_2 - p_1 = 4 - 6 \)

\[ = -2 \]

Percentage change in price \( p_2 - p_1 / p_2 + p_1/2 \times 100 \)

\[ -2/10/2 \times 100 \]

\[ = -40 \]
Price elasticity of demand = % change in quantity demanded / % change in price

\[ = \frac{40}{-40} \]

\[ = -1 \]

We ignore the minus sign; therefore price elasticity of demand is equal to one.

1.33 Income Elasticity

Income elasticity of demand shows the degree of responsiveness of quantity demanded of good to a small change in income of consumers. The degree of response of quantity demanded to a change in income is measured by dividing the proportionate change in quantity demanded by the proportionate change in income. Thus, more precisely, the income elasticity of demand may be defined as the ratio of the proportionate change in the quantity purchased of a good to the proportionate change in income which induce the former.

The following are the three propositions

1. If proportion of income spent on the good remains the same as income increases, then income elasticity for the good is equal to one.

2. If proportion of income spent on the good increases as income increases, then the income elasticity for the good is greater than one.

3. If proportion of income spent on the good decreases as income rises then income elasticity for the good is less than one.

Income elasticity

\[ E_i = \frac{Proportionate \ change \ in \ quantity \ purchased \ of \ a \ good}{Proportionate \ change \ in \ income} \]

\[ = \frac{Q/Q}{M/M} \]

\[ = \frac{Q/Q \times M/M}{M/Q} \]

Let, \( M \) stands for an initial income, \( M \) for a small change in income, \( Q \) for the initial quantity purchased. \( Q \) for a change in quantity purchased as a result of a change in income and \( E_i \) for income elasticity of demand.

Midpoint formula for measuring income elasticity of demand when changes in income are quite large can be written as.

\[ E_i = \frac{Q_2 - Q_1/Q_2 + Q_1/2 - M_2 - M_1/M_2 + M_1/2}{M_2 - M_1/M_2 + M_1/2} \]

\[ = \frac{Q_2 + Q_1}{M_2} \times M_1/M_2 \]

\[ = \frac{Q}{M} \times M_2 + M_1/Q_2 + Q_1} \]
Example: If a consumer daily income rises from Rs. 300 to Rs. 350, his purchase of a good X increases from 25 units per day to 40 units; find the income elasticity of demand for X?

Change in quantity demand (Q) = Q₂ - Q₁
= 40 - 25
= 15

Change in income (M) = M₂ - M₁
= 350 - 300
= 50

\[ E_i = \frac{\% \text{ change in quantity demanded}}{\% \text{ change in price}} \]
\[ = \frac{\Delta Q}{\Delta M} \times \frac{M₂}{Q₂} + \frac{M₁}{Q₁} \]
\[ = \frac{15}{50} \times \frac{350}{25} + \frac{300}{40} \]
\[ = \frac{15}{30} \times \frac{650}{65} \]
\[ = 3 \]

Income elasticity of demand in this case is 3.

\[ E_i = 0 \] implies that a given increase in income does not at all lead to any increase in quantity demanded of the good or increase in expenditure on it. (It signifies that quantity demanded of the good is quite unresponsive to change in income.

\[ E_i > 0 \] (ie +ve), then an increase in income lead to the increase in quantity demanded of the good. This happens in case of normal or superior goods.

\[ E_i < 0 \] (is –ve), in such cases increase in income will lead to the fall in quantity demanded of the goods. Goods having negative income elasticity are known as inferior goods.

1.34 Cross elasticity of demand

Very often demand for the two goods are so related to each other that when the price of any of them changes, the demand for the other goods also changes, when its own price remains the same. Therefore, the change in the demand for one good represents the cross elasticity of demand of one good for the other.

Cross elasticity of demand of X for Y = \[ \frac{\text{Proportionate change in quantity demanded by X}}{\text{Proportionate change in the price of good Y}} \]

Or

\[ E_c = \frac{q_X}{q_Y} / \frac{p_Y}{p_Y} \]
\[ \frac{\Delta q_x}{q_x} \times \frac{p_y}{\Delta p_y} = \frac{\Delta q_x}{\Delta p_y} \times \frac{p_y}{q_x} \]

Where, \( Ec \) stands for cross elasticity of \( X \) for \( Y \)
- \( q_x \) stands for the original quantity demanded of \( X \)
- \( \Delta q_y \) stands for changes in quantity demanded of good \( X \)
- \( p_y \) stands for original price of good \( X \)
- \( \Delta p_y \) stands for small changes in the price of \( Y \)

When change in price is large, we would use midpoint method for estimating cross elasticity of demand. Note that when we divide percentage change in quantity demanded by percentage change in price, 100 in both numerator and denominator for cancel out. Therefore, we can write midpoint formula for measuring cross elasticity of demand as:

\[ Ec = \frac{q_x_2 - q_x_1}{q_x_2 + q_x_1/2} \div \frac{p_y_2 - p_y_1}{p_y_2 + p_y_1/2} \]

**Example**: If the price of coffee rises from Rs. 45 per 250 grams per pack to Rs. 55 per 250 grams per pack and as a result the consumers demand for tea increases from 600 packs to 800 packs of 250 grams, then the cross elasticity of demand of tea for coffee can be found out as follows.

Hints: we use midpoint method to estimate cross elasticity of demand.

Change in quantity demanded of tea = \( q_{t_2} - q_{t_1} \)

\[ = 800 - 600 \]
\[ = 200 \]

Change in price of coffee = \( p_{c_2} - p_{c_1} \)

\[ = 55 - 45 \]
\[ = 10 \]

Substituting the values of the various variables in the cross elasticity formula, we have

Cross elasticity of demand = \( \frac{800 - 600/800 + 600/2}{55 - 45/55 + 45/2} \)

\[ = \frac{200/700 \times 50/10}{10/7} \]
\[ = 1.43 \]
In the example of tea and coffee, above, when two goods are substitutes of each other, then as a result of the rise in price of one good, the quantity demanded of the other good increases. Therefore, the cross elasticity of demand between the two substitute good is positive, that is, in response to the rise in price of one good, the demand for the other good rises. Substitute goods are also known as competing goods. On the other hand, when the two good are complementary with each other just as bread and butter, tea and milk etc..., rise in price of one good bring about the decrease in the demand for the two complimentary good is negative. Therefore, according to the classification based on the concept of cross elasticity of demand, good X and good Y are substitute or complements according as the cross elasticity of demand is positive or negative.

**Example:** Suppose the following demand function for coffee in terms of price of tea is given. Find out the cross elasticity of demand when price of tea rises from Rs. 50 per 250 grams pack to Rs. 55 per 250 grams pack.

\[
Q_c = 100 + 205 Pt
\]

When \(Q_c\) is the quantity demanded of coffee in terms of pack of 250 grams and \(Pt\) is the price of tea per 250 grams pack.

**Solution:** The +ve sign of the coefficient of \(Pt\) shows that rise in price of tea will cause an increase in quantity demanded of coffee. This implies that tea and coffee are substitutes.

The demand function equation implies that coefficient:

\[
\frac{\partial Q_c}{\partial Pt} = 2.5
\]

In order to determine cross elasticity of demand between tea and coffee, we first find out quantity demanded of coffee when price of tea is Rs.50 per 250 grams. Thus,

\[
Q = 100 + 2.5 \times 50
\]

\[
= 225
\]

Cross elasticity, \(Ce\) = \(\frac{\partial Q_c}{\partial Pt} \times Pt/Qc\)

\[
= 2.5 \times \frac{50}{225}
\]

\[
= 125/225
\]

\[
= 0.51
\]

### 1.35 Engel Function

An Engel function shows the relationship between quantity demanded of a good and level of consumer’s income. Since with the increase in income normally more quantity of the good is demanded, Engel curve slope upward (i.e. it has a positive slope). Although the Engel curve for normal goods slopes upward but it is of different shape for different goods. It is convex or concave, depending on whether the good is a necessity or a luxury. In case of an inferior good for which
income effect is negative, that is, less is demanded when income raises, Engel curve is backward bending.

**Exercises:**

1. If the demand Law is given by \( q = \frac{20}{p+1} \), find the elasticity of demand with respect to price at the point when \( p = 3 \).

   Solution: Elasticity of demand = \(-\frac{\Delta q}{\Delta p} \times \frac{p}{q}\)

   \[ q = 20 \left( \frac{1}{p + 1} \right) \]

   \[ \frac{\Delta q}{\Delta p} = -20 \left( \frac{1}{p + 1} \right)^2 \]

   When \( p = 3 \),

   \[ q = \frac{20}{4} = 5 \]

   \[ \frac{\Delta q}{\Delta p} = -\frac{20}{16} = -\frac{5}{4} \]

   Elasticity of demand = \(\frac{5}{4} \times \frac{3}{5} = \frac{3}{4}\)

2. The demand function \( p = 50 - 3x \), when \( p = 5 \), then \( 5 = 50 - 3x \)

   \[ X = 15 \]

   \[ \frac{d}{dp}(p) = \frac{d}{dp}(50 - 3x) \]

   \[ 1 = -3 \times \frac{dx}{dp} \]

   or

   \[ \frac{dx}{dp} = -\frac{1}{3} \]

   \[ e = \frac{dx}{dp} \times \frac{p}{x} \]

   \[ = \frac{5}{15} \times \frac{1}{3} = \frac{1}{9} \]
3. The total cost $C(x)$ associated with producing and marketing $x$ units of an item is given by $C(x) = 0.005 x^3 - 0.02 x^2 - 30x + 3000$. Find

(1) Total cost when output is 4 units.

(2) Average cost of output of 10 units.

Solution: (1) Given that $C(x) = 0.005 x^3 - 0.02 x^2 - 30x + 3000$.

For $x = 4$ units, the total cost $C(x)$ becomes

$$C(x) = 0.005(4)^3 - 0.02(4)^2 - 30 	imes 4 + 3000$$

$$= 0.32 - 0.32 - 120 + 3000$$

$$= Rs. 2880$$

(2) Average cost ($AC$) = $TC/x$

$$= \frac{0.005 x^3 - 0.02 x^2 - 30x + 3000}{x}$$

$$= 0.005 x^2 - 0.02x - 30$$

4. The utility function of the consumer is given by $U = x_1x_2^2 - 10x_1$

Where $x_1$ and $x_2$ are the quantities of two commodities consumed. Find the optimal utility value if his income is 116 and product prices are 2 and 8 respectively.

Solution: We have the utility function $U = x_1x_2^2 - 10x_1$, and the

Budget constraint $116 - 2x_1 - 8x_2 = 0$,

From the budget equation we get:

$$x_1 = 58 - 4x_2$$

$$U = (58 - 4x_2)x_2^2 - 10(58 - 4x_2)$$

$$= 58x_2^2 - 4x_2^2 - 580 - 40x_2$$

For minimum utility $\frac{dU}{dx_2} = 0$

$$\frac{dU}{dx_2} = 116x_2 - 12x_2^2 - 40$$

$$= 3x_2^2 - 29x_2 - 10 = 0$$

$$= 3x_2^2 - 3x_2 + x_2 - 10 = 0$$
\[ = 3 \ x_2( x_2 - 10) + (x_2 -10) = 0 \]

\[(3 \ x_2 + 1)( x_2 - 10) = 0 \]

Or \( x_2 = -1/3 \) or \( x_2 = 10 \) \( x_2 \) cannot be negative

Hence \( x_2 = 10 \)

\[ \frac{d^2u}{dx^2} = 116 - 24 \ x_2 \]

\[ = 116 - 240 \]

\[ = -124 < 0 \] Maxima is confirmed.

\( x_2 = 10, \)

\( x_1 = 58 - 40 \)

\[ = 18 \]

5. A function \( p = 50 - 3x \), find TR, AR, MR

\( TR = P \times x \)

\[ = (50 - 3x) \times \]

\[ = 50x - 3x^2 \]

\( AR = TR/x \)

\[ = 50x - 3x^2 \]

\[ = 50 - 3x \]

\( MR = \frac{d(50x-3x^2)}{dx} \)

\[ = \frac{d(50x)}{dx} - \frac{d(3x^2)}{dx} \]

\[ = 50x - 6x \]

6. The following production function is given:

\[ Q = L^{0.75}K^{0.25} \]

Find the marginal product of labour and marginal product of capital

\[ MP_L = \frac{\partial Q}{\partial L} \]
7. The demand function for mutton is:

\[ Q_M = 4850 - 5P_M + 1.5P_c + .1Y \]

Find the income elasticity of demand and cross elasticity of demand for mutton. \( Y \) (income) = Rs.1000, \( P_M \) (price of mutton) = Rs.200, \( P_c \) (price of chicken) Rs.100.

**Solution:** income elasticity = \( \frac{\partial Q}{\partial Y} \times \frac{Y}{Q} \)

\[ \frac{\partial Q}{\partial Y} = .1 \text{ income elasticity} = (.1) \times 1000 \]

\[ = 4850 - 1000 + 150 + 100 \]

\[ = 5100 - 1000 \]

\[ = 4100 \]

Income elasticity = \( \frac{.1 \times 1000}{4100} \)

\[ = \frac{1}{41} \]

Cross elasticity of mutton = \( \frac{dq_m}{dp_c} \times \frac{p_c}{q_m} \)

\[ = \frac{dq_m}{dp_c} \]

\[ = 1.5, \text{ cross elasticity} \]

\[ = 1.5 \times \frac{100}{4100} \]

\[ = \frac{1.5}{41} \]

\[ = \frac{3}{2 \times 41} \]
8. Find the elasticity of supply for supply function \( x = 2p^2 + 5 \) when \( p = 3 \)

\[
\text{Es} = \frac{p}{x} \times \frac{dx}{dp} = \frac{p}{x} \times 4p = \frac{4 \times 3}{18 + 5} = \frac{36}{23}
\]

9. The consumption function for an economy is given by \( c = 50 + .4y \), find MPC and MPS

Here MPC = .4

\[
\text{MPS} = 1 - \text{MPC} = 1 - .4 = .6
\]

10. For a firm, given that \( c = 100 + 15x \) and \( p = 3 \), find profit function

Profit function \( \pi = TR - TC \)

\[
\text{TR} = p \times x = 3x
\]

\[
\Pi = 3x - 100 + 15x = 18x -100
\]

11. The demand function \( p = 50 - 3x \), find MR

\[
\text{TR} = P \times X = (50 - 3x) \times x = 50x - 3x^2
\]

\[
\text{MR} = \frac{\text{dTR}}{dX} = 50 - 6x
\]
12. The total cost function is \( TC = 60 - 12x + 2x^2 \). Find the MC

\[
MC = \frac{dT C}{dx}
\]

\[= 12 + 4x\]

**FURTHER READINGS**

5. R.G.D Allen- Mathematical Economics.
MODULE II

CONSTRAINT OPTIMIZATION, PRODUCTION FUNCTION AND LINEAR PROGRAMMING


2.1 Constraint Optimization Methods

The problem of optimization of some quantity subject to certain restrictions or constraint is a common feature of economics, industry, defense, etc. The usual method of maximizing or minimizing a function involves constraints in the form of equations. Thus utility may be maximized subject to the budget constraint of fixed income, given in the form of a equation. The minimization of cost is a familiar problem to be solved subject to some minimum standards. If the constraints are in the form of equations, methods of calculus can be useful. But if the constraints are inequalities instead of equations and we have an objective function to be optimized subject to these inequalities, we use the method of mathematical programming.

2.2 Substitution Method

Another method of solving the objective function with subject to the constraint is substitution methods. In this method, substitute the values of x or y, and the substitute this value in the original problem, differentiate this with x and y.

Consider a utility function of a consumer

\[ U = x^{0.3} + y^{0.3}, \]

The budget constraint \(20x + 10y = 200.\)

Rewrite the above equation

\[ y = \frac{200 - 20x}{10} \]

\[ = 20 - 2x \]

Then the original utility function \(U = x^{0.3} + (20 - 2x)\)

2.3 Lagrange Method

Constrained maxima and minima: In mathematical and economic problems, the variables in a function are sometimes restricted by some relation or constraint. When we wish to maximize or minimize \(f(x_1, x_2...x_n)\) subject to the condition or constraint \(g(x_1, x_2...x_n) = 0\), there exist a method known as the method of Lagrange Multiplier. For example utility function \(U = u(x_1, x_2...x_n)\) may be subject to the budget constraint that income equals expenditure that is \(Y = p_1x_1 + p_2x_2 + \ldots + p_nx_n\). We introduce a new variable \(\lambda\) called the Lagrange Multiplier and construct the function.
\[ Z = f(x_1, x_2 \ldots x_n) + \gamma g(x_1, x_2 \ldots x_n) \]

This new function \( z \) is a function on \( n + 1 \) variable \( x_1, x_2 \ldots x_n \) and \( \gamma \)

**Example:** Given the utility function \( U = 4x_1^{1/2}x_2^{1/2} \) and budget constraint \( 60 = 2x_1 + x_2 \).

Find the condition for optimality.

(Hint. \( U = 4x_1^{1/2}x_2^{1/2} \))

\[
\begin{align*}
\frac{\partial F}{\partial x_1} &= 4x_2^{1/2} \cdot 1/2 x_1^{-1/2} - 2\gamma = 0 \\
\frac{\partial F}{\partial x_2} &= 4x_1^{1/2} \cdot 1/2 x_2^{-1/2} - \gamma = 0 \\
\frac{\partial F}{\partial \gamma} &= 60 - 2x_1 - x_2 = 0 \\
&= 4x_2^{1/2,1/2} x_1^{-1/2} / 4x_1^{1/2,1/2} x_2^{-1/2} \\
&= \frac{2\gamma}{\gamma} \\
&= \frac{x_2}{x_1} \\
&= \frac{2}{1} \\
&= 2x_2
\end{align*}
\]

Or

\[ = x_2 \]

**2.4 Utility Maximization**

There are two approaches to study consumer behavior- the first approach is a classical one and is known as cardinal utility approach and the second approach is ordinal utility approach popularly known as indifference curve approach. In both the approaches, we assume that consumer always behaves in a rational manner, because he derives the maximum utility (satisfaction) out of his budget constraint.

**Example:** The utility function of the consumer is given by \( u = x_1x_2^2 - 10x_1 \) where \( x_1 \) and \( x_2 \) are the quantities of two commodities consumed. Find the optimal utility value if his income is 116 and product prices are 2 and 8 respectively.

**Solution:** we have utility function
U = f (x₁, x₂) = u = x₂x₂² - 10x₁, and

Budget constraint 116 – 2x₁ - 8x₂ = 0 from budget equation we get x₁ = 58 – 4x₂

U = (58 – 4x₂)x₂² - 10(58 -4x₂)
= 58x₂² - 4x₂³ - 580 – 40x₂

For minimum utility

\[ \frac{du}{dx₂} = 0 \]

= 116x₂ - 12x₂² - 40 = 0

3x₂² - 29x₂ - 10 = 0
3x₂² - 30x₂ + x₂ - 10 = 0
3x₂(x₂ - 10) + (x₂ - 10) = 0
(3x₂ + 1)(x₂ - 10) = 0

Or

x₂ = -1/3

Or

x₂ = 10x₂ cannot be negative.

Hence x₂ = 10

\[ \frac{d^2u}{dx₂²} = 116 - 24x₂ \]

= 116 - 240

= -124 ≺ 0 maxima is confirmed.

x₂ = 10 ,

x₁ = 58 – 40
= 18

We know that consumer’s equilibrium (condition of maximum utility) is fulfilled when

\[ \frac{MU₁}{MU₂} = \frac{p₁}{p₂} \]

MU₁ = x₂² - 10
\[ MU_2 = 2x_2x_1 \]

\[ \frac{MU_2}{MU_1} = \frac{e^{0.5x-10}}{2x_2x_1} \]

\[ = \frac{100-10}{2 \times 15 \times 10} \]

\[ = \frac{90}{360} \]

\[ = \frac{1}{4} \]

Hence maximum utility is obtained when \( x_1 = 18 \) and \( x_2 = 10 \)

### 2.5 Cost Minimization

Cost minimization involves how a firm has to produce a given level of output with minimum cost. Consider a firm that uses labour (L) and capital (K) to produce output (Q). Let W is the price of labour, that is, wage rate and \( r \) is the price of capital and the cost (C) incurred to produce a level of output is given by

\[ C = wL + rK \]

The objective of the form is to minimize cost for producing a given level of output. Let the production function is given by following.

\[ Q = f (L, K) \]

In general there is several labour – capital combinations to produce a given level of output. Which combination of factors a firm should choose which will minimize its total cost of production. Thus, the problem of constrained minimization is

\[ \text{Minimize } C = wL + rK \]

Subject to produce a given level of output, say \( Q_1 \) that satisfies the following production function

\[ Q_1 = f (L, K) \]

The choice of an optimal factor combination can be obtained through using Lagrange method.

Let us first form the Lagrange function is given below

\[ Z = wL + rK + \gamma(Q_1- f (L, K)) \]

Where \( \gamma \) is the lagrange multiplier

For minimization of cost it necessary that partial derivatives of \( Z \) with respect to \( L, K \) and \( \gamma \) be zero
Note that \( \frac{\partial f(L, K)}{\partial L} \) and \( \frac{\partial f(L, K)}{\partial K} \) are the marginal physical products of labour and capital respectively.

Rewriting the above equation we have

\[
\begin{align*}
\frac{\partial z}{\partial l} &= w - \gamma \frac{\partial f(L, K)}{\partial L} = 0 \\
\frac{\partial z}{\partial K} &= r - \gamma \frac{\partial f(L, K)}{\partial K} = 0 \\
\frac{\partial z}{\partial y} &= Q_1 - f(L, K) = 0
\end{align*}
\]

The last equations shows that total cost is minimized when the factor price ratio \( \frac{w}{r} \) equal the ratio of MPP of labour and capital.

### 2.6 Profit Maximization

Maximizations of profit subject to the constraint can also used to identify the optimum solution for a function.

Assumes that \( TR = PQ \)

\[
TC = wL + rK
\]

\[
\Pi = TR - TC
\]

\[
\Pi = PQ - (wL + rK)
\]

Thus the objective function of the firm is to maximize the profit function

\[
\Pi = PQ - (wL + rK)
\]

The firm has to face a constraint \( Q = f(K, L) \)

From the Lagrange function,

\[
Z = (PQ - (wL + rK)) + \gamma (F(K, L) - Q)
\]

\[
\frac{\partial Z}{\partial Q} = P - \gamma = 0 \quad \therefore \quad \gamma = \frac{P}{\gamma} \quad (1)
\]
From equation (1), we get \( P = \gamma \)

Substitute this in (2) and (3)

From (2)
\[
\frac{\partial z}{\partial L} = -w + \gamma fL = 0 \quad \text{(2)}
\]
\[
\frac{\partial z}{\partial K} = -r + \gamma fK = 0 \quad \text{(3)}
\]
\[
\frac{\partial z}{\partial Q} = f(K, L) - Q = 0 \quad \text{(4)}
\]

From (2)
\[
w = \gamma fL
\]

Substituting \( P = \gamma \)

We get
\[
w = PfL \quad \text{(5)}
\]

Equation (3)
\[
r = PfK \quad \text{(6)}
\]

Now in (5)
\[
P = \frac{w}{fL}
\]

And in (6) \( P = \frac{r}{fK} \) rewrite the above equation
\[
\frac{W}{fL} = \frac{r}{fK}
\]

Cross multiplying
\[
w/r = fL/fK
\]

The above condition is profit maximization

2.7 Production Function

Production function is a transformation of physical inputs into physical outputs. The output is thus a function of inputs. The functional relationship between physical inputs and physical output of a firm is known as production function. Algebraically, production function can be written as,

\[
Q = f(a, b, c, d, \ldots)
\]

Where, \( Q \) stands for the quantity of output, \( a, b, c, d, \) etc; \( Q \) stands for the quantitative factors. This function shows that the quantity \( q \) of output produced depends upon the quantities, \( a, b, c, d \) of the factors \( A, B, C, D \) respectively.

The general mathematical form of Production function is:

\[
Q = f(L, K, R, S, v, e)
\]
Where: Q stands for the quantity of output, L is the labour, K is capital, R is raw material, S is the Land, v is the return to scale and e is efficiency parameters.

According to G.J. Stigler, “the production function is the name given the relationship between the rates of inputs of productive services and the rates of output of product. It is the economists summary of technological knowledge. Thus, production function express the relationship between the quantity of output and the quantity of various input used for the production. More precisely the production function states the maximum quantity of output that can be produced from any given quantities of various inputs or in other words, if stands the minimum quantities of various inputs that are required to yield a given quantities of output.

“Production function of the firm may also be derived as the minimum quantities of wood, varnish, labour time, machine time, floor space, etc; that are required to produce a given number of table per day”.

Knowledge of the production function is a technological or engineering knowledge and is provided to the form by its engineers or production managers. Two things must be noted in respect of production function. First, production functions like demand function, must be considered with reference to a particular period of time. Production function expresses flows of inputs resulting in flows of output in a specific period of time. Secondly, production function of a firm is determined by the state of technology. When there is advancement in technology, the production function charges with the result that the new production function charges with the result of output from the given inputs, or smaller quantities of inputs can be used for producing a given quantity of output.

2.7 Linear, Homogeneous Production Function

Production function can take several forms but a particular form of production function enjoys wide popularity among the economists. This is a linear homogeneous production function, that is, production function which is homogenous production function of the first degree. Homogeneous production function of the first degree implies that if all factors of production are increased in a given proportion, output also increased in a same proportion. Hence linear homogeneous production function represents the case of constant return to scales. If there are two factors X and Y, The production function and homogeneous production function of the first degree can be mathematically expressed as,

\[ Q = f(X, Y) \]

Where Q stands for the total production, X and Y represent total inputs.

\[ mQ = f(mX, mY) \]

m stands any real number

The above function means that if factors X and y are increased by m-times, total production Q also increases by m-times. It is because of this that homogeneous function of the first degree yield constant return to scale.
More generally, a homogeneous production function can be expressed as

\[ Q_{mk} = (mX, mY) \]

Where \( m \) is any real number and \( k \) is constant. This function is homogeneous function of the \( k^{th} \) degree. If \( k \) is equal to one, then the above homogeneous function becomes homogeneous of the first degree. If \( k \) is equal to two, the function becomes homogeneous of the 2\(^{nd}\) degree.

If \( k > 1 \), the production function will yield increasing return to scale.

If \( k < 1 \), it will yield decreasing return to scale.

2.8 Fixed Proportion Production Function

Production function is of two qualitatively different forms. It may be either fixed-proportion production function or variable proportion production functions. Whether production function is of a fixed proportion form or a variable proportion form depends upon whether technical coefficients of production are fixed or variable. The amount of a productive factor that is essential to produce a unit of product is called the technical coefficient of production. For instance, if 25 workers are required to produce 100 units of a product, then 0.25 is the technical coefficient of labour for production. Now, if the technical coefficient of production of labour is fixed, then 0.25 of labour unit must be used for producing a unit of product and its amount cannot be reduced by using in its place some other factor. Therefore, in case of fixed proportions production function, the factor or inputs, say labour and capital, must be used in a definite fixed proportion in order to produce a given level of output. A fixed proportion production function can also be illustrated by equal product curve or isoquants. As in fixed proportion production function, the two factors, say capital and labour, must be used in fixed ratio, the isoquants of such a production function are right angled.

Suppose in the production of a commodity, capital-labour ratio that must be used to produce 100 units of output is 2:3. In this case, if with 2 units of capital, 4 units of labour are used, then extra one unit of labour would be wasted; it will not add to total output. The capital-labour ratio must be maintained whatever the level of output.

If 200 units of output are required to be produced, then, given the capital-output ratio of 2:3, 4 units of capital and 6 units of labour will have to be used.

If 300 units of output are to be produced, then 6 units of labour and 9 units labour will have to be used.

Given the capital–labour ratio of 2:3, an isoquant map of fixed–proportion production function has been drawn in the given figure.
In a fixed proportion production function, doubling the quantities of capital and labour at the required ratio doubles the output, trebling their quantities at the required ratio trebles the output.

2.9 Cobb – Douglas Production Function

Many Economists have studied actual production function and have used statistical methods to find out relations between changes in physical inputs and physical outputs. A most familiar empirical production function found out by statistical methods is the Cobb – Douglas production function. Cobb – Douglas production function was developed by Charles Cobb and Paul Douglas. In C-D production function, there are two inputs, labour and capital, Cobb – Douglas production function takes the following mathematical form

$$Q = AL^α K^β$$

Where Q is the manufacturing output, L is the quantity of labour employed, K is the quantity of capital employed, A is the total factor productivity or technology are assumed to be a constant. The α and β, output elasticity’s of Labour and Capital and the A,α and β are positive constant.

Roughly speaking, Cobb –Douglas production function found that about 75% of the increasing in manufacturing production was due to the Labour input and the remaining 25 % was due to the Capital input.

2.10 Properties of Cobb – Douglas Production Function

2.10.1. Average product of factors: The first important properties of C – D production function as well as of other linearly homogeneous production function is the average and marginal products of factors depend upon the ratio of factors are combined for the production of a commodity. Average product if Labour (APL) can be obtained by dividing the production function by the amount of Labour L. Thus,
Average Product Labour (Q/L)

\[ Q = AL^\alpha K^\beta \]

\[ \frac{Q}{L} = \frac{AL^\alpha K^\beta}{L} = AK^\beta L^{1-\alpha} = A \left( \frac{K}{L} \right)^\beta \]

Thus Average Product of Labour depends on the ratio of the factors (K/L) and does not depend upon the absolute quantities of the factors used.

Average Product of Capital (Q/K)

\[ Q = AL^\alpha K^\beta \]

\[ \frac{Q}{K} = \frac{AL^\alpha K^\beta}{K} = AL^\alpha K^{\beta-1} = A \left( \frac{L}{K} \right)^\alpha \]

So the average Product of capital depends on the ratio of the factors (L/K) and does not depend upon the absolute quantities of the factors used.

2.10.2 Marginal Product of Factors: The marginal product of factors of a linear homogenous production function also depends upon the ratio of the factors and is independent of the absolute quantities of the factors used. Note, that marginal product of factors, says Labour, is the derivative of the production function with respect to Labour.

\[ Q = AL^\alpha K^\beta \]

\[ MP_L = \frac{\partial Q}{\partial L} = \alpha AL^\alpha K^{\beta-1} = A\alpha K^\beta L^{1-\alpha} = \alpha A \left( \frac{K}{L} \right)^\beta \]

\[ MP_L = \alpha AP_L \]

It is thus clear that MP_L depends on capital –labour ratio, that is, Capital per worker and is independent of the magnitudes of the factors employed.
\[ Q = AL^\alpha K^\beta \]

\[ MP_K = \frac{\partial Q}{\partial K} \]
\[ = \beta K^{\beta-1} L^\alpha \]
\[ = \beta AL^\alpha / K^{1-\beta} \]
\[ = \beta A \left( \frac{L}{K} \right)^\alpha \]
\[ MP_L = \beta AP_K \]

It is thus clear that MPL depends on capital – labour ratio, that is, capital per worker and is independent of the magnitudes of the factors employed.

2.10.3 Marginal rate of substitution: Marginal rate of substitution between factors is equal to the ratio of the marginal physical products of the factors. Therefore, in order to derive MRS from Cobb–Douglas production function, we used to obtain the marginal physical products of the two factors from the C – D function.

\[ Q = AL^\alpha K^\beta \]

Differentiating this with respect to L, we have

\[ MP_L = \frac{\partial Q}{\partial L} \]
\[ = \partial AL^\alpha K^\beta \]
\[ = \alpha AL^{\alpha-1} K^\beta \]
\[ = \alpha (AL^\alpha K^\beta) \]

Now, \( Q = AL^\alpha K^\beta \), Therefore,
\[ \frac{dQ}{dL} \text{ Represents the marginal product of labour and } \frac{Q}{L} \text{ stands for the average of labour.} \]

Thus, \( MP_L = \alpha (AP_L) \)

Similarly, by differentiating C –D production function with respect to capital, we can show that marginal product of capital

\[ Q = AL^\alpha K^\beta \]

\[ MP_K = \frac{\partial Q}{\partial K} \]

\[ = \beta AL^\alpha K^{\beta-1} \]

\[ = \beta \left( \frac{AL^\alpha K^\beta}{K'} \right) \]

\[ = \beta \left( \frac{Q}{K} \right) \]

\[ MRS_{LK} = \frac{MP_L}{MP_K} \]

\[ = \frac{\alpha (\frac{Q}{L})}{\beta (\frac{Q}{K})} \]

\[ = \frac{\alpha}{\beta} \times \frac{K}{L} \]

1. C –D production function and Elasticity of substitution (\( \ell_s \) or \( \sigma \)) is equal to unity.

\[ \ell_s = \frac{\text{Proportionate change in Capital-Labour ratio} \left( \frac{K}{L} \right)}{\text{Proportionate change in } MRS_{LK}} \]

\[ = \frac{\left( \frac{d(K)}{K/L} \right)}{\left( \frac{d \left( \frac{MRS}{K/L} \right)}{MRS_{LK}} \right)} \]

Substituting the value of MRS obtain in above
2.10.4 Return to Scale: An important property of C–D production function is that the sum if its exponents measures returns to scale. That is, when the sum of exponents is not necessarily equal to zero is given below.

\[ Q = AL^\alpha K^\beta \]

In this production function the sum of exponents \((\alpha + \beta)\) measures return to scale. Multiplying each input labour \((L)\) and capital \((K)\), by a constant factor \(g\), we have

\[ Q' = A(gL)^\alpha (gK)^\beta \]

\[ = g^\alpha g^\beta (AL^\alpha K^\beta) \]

\[ = g^{\alpha + \beta} (AL^\alpha K^\beta) \]

\[ \text{I.e.} \quad Q' = g^{\alpha + \beta} Q \]

This means that when each input is increased by a constant factor \(g\), output \(Q\) increases by \(g^{\alpha + \beta}\). Now, if \(\alpha + \beta = 1\) then, in this production function.

\[ Q' = gQ \]

\[ Q' = gQ \]

This is, when \(\alpha + \beta = 1\), output \((Q)\) also increases by the same factor \(g\) by which both inputs are increased. This implies that production function is homogeneous of first degree or, in other words, return to scale are constant.

When \(\alpha + \beta > 1\), say it is equal to 2, then, in this production function new output.

\[ Q' = g^{\alpha + \beta} AL^\alpha K^\beta \]

\[ = g^2Q. \]
In this case multiplying each input by constant $g$, then output ($Q$) increases by $g^2$. Therefore, $\alpha + \beta > 1$.

C –D production function exhibits increasing return to scale. When $\alpha + \beta < 1$, say it is equal to 0.8, then in this production function, new output.

$$Q' = g^{\alpha + \beta} A L^\alpha K^\beta$$

$$= g^{0.8} Q.$$

That is increasing each input by constant factor $g$ will cause output to increase by $g^{0.8}$, that is, less than $g$. Return to scale in this case are decreasing. Therefore $\alpha + \beta$ measures return to scale.

If $\alpha + \beta = 1$, return to scale are constant.

If $\alpha + \beta > 1$, return to scale are increasing.

If $\alpha + \beta < 1$, return to scale are decreasing.

2.10.5 C-D Production Functions and Output Elasticity of Factors

The exponents of labour and capital in C –D production function measures output elasticity’s of labour and capital. Output elasticity of a factor refers to the relative or percentage change in output caused by a given percentage change in a variable factor, other factors and inputs remaining constant. Thus,

$$O E = \frac{\partial Q}{\partial L} \times \frac{L}{Q}$$

$$= a \times \frac{Q}{L} \times \frac{L}{Q}$$

$$= a$$

Thus, exponent ($a$) of labour in C –D production function is equal to the output elasticity of labour.

Similarly, O E of Capital $= \frac{\partial Q}{\partial K} \times \frac{K}{Q}$

$$\text{MPK} = b \times \frac{K}{Q}$$

$$= b \times \frac{Q}{K} \times \frac{K}{Q}$$

$$= b$$

Therefore, output elasticity of capital $= b \times \frac{Q}{K} \times \frac{K}{Q}$

$$= b$$. 
2.10.6 C–D production Function and Euler’s theorem

C–D production function \( Q = AL^aK^b \)

Where \( a + b = 1 \) helps to prove Euler theorem. According to Euler theorem, total output \( Q \) is exhausted by the distributive shares of all factors when each factor is paid equal to its marginal physical product. As we know

\[
\begin{align*}
MP_L &= A a \left( \frac{K}{L} \right)^b \\
MP_K &= A a \left( \frac{L}{K} \right)^a
\end{align*}
\]

According to Euler’s theorem if production functions is homogeneous of first degree then, Total output, \( Q = L \times MP_L + K \times MP_K \), substituting the values of \( MP_L \) and \( MP_K \), we have

\[
Q = L \times Aa \left( \frac{K}{L} \right)^b + K \times Aa (L/K)^a
\]

\[
= Aa \left( \frac{L}{K} \right)^{1-b} K^b + AbL^a K^{1-a}
\]

Now, in C–D production function with constant return to scale \( a + b = 1 \) and

Therefore: \( a = 1 - b \) and \( b = 1 - a \), we have

\[
Q = Aa L^a K^b + Ab L^a K^b
\]

\[
= (a + b) A L^a K^b
\]

Since \( a + b = 1 \) we have

\[
Q = A L^a K^b
\]

\[
Q = Q
\]

Thus, in C–D production function with \( a + b = 1 \) if wage rate = \( MP_L \) and rate of return on capital (K) = \( MP_K \), then total output will be exhausted.

2.10.7 C–D Production Function and Labour Share in National Income.

C–D production function has been used to explain labour share in national income (i.e., real national product). Let \( Y \) stand for real national product, \( L \) and \( K \) for inputs of labour and capital, then according to C–D production function as applied to the whole economy, we have

\[
Y = AL^a K^{1-a} \quad \text{(1)}
\]

Now, the real wage of labour (w) is its real marginal product. If we differentiate \( Y \) partially with respect to \( L \), we get the marginal product of labour, thus, Real wage (or marginal product of labour)

\[
W = \frac{dY}{dL}
\]
Total wage bill = \( wL = \frac{dY}{dL} = aAL^aK^{1-a} \) ........ (2)

From (1) and (2), we get,

The labour share in real national product

\[
\frac{Total \ wage \ bill}{Real \ national \ product} = \frac{wL}{Y} = \frac{aAL^aK^{1-a}}{AL^aK^{1-a}} = a
\]

Thus, according to C – D production function, labour’s share in real national product will be a constant ‘a’ which is independent of the size of labour force.

2.11 Linear Programming Problems (LPP)

The term linear programming consists of two words, linear and programming. The linear programming considers only linear relationship between two or more variables. By linear relationship we mean that relations between the variable can be represented by straight lines. Programming means planning or decision-making in a systematic way. “Linear programming refers to a technique for the formulation and solution of problems in which some linear function of two or more variables is to be optimized subject to a set of linear constraints at least one of which must be expressed as inequality”. American mathematician George B. Danzig, who invented the linear programming technique.

Linear programming is a practical tool of analysis which yields the optimum solution for the linear objective function subject to the constraints in the form of linear inequalities. Linear objective function and linear inequalities and the techniques, we use is called linear programming, a special case of mathematical programming.

2.12 Terms of Linear Programming

(1) Objective Function

Objective function, also called criterion function, describe the determinants of the quantity to be maximized or to be minimized. If the objective of a firm is to maximize output or profit, then this is the objective function of the firm. If the linear programming requires the minimization of cost, then this is the objective function of the firm. An objective function has two parts – the primal and dual. If the primal of the objective function is to maximize output then its dual will be the minimization of cost.
(2) Technical Constraints

The maximization of the objective function is subject to certain limitations, which are called constraints. Constraints are also called inequalities because they are generally expressed in the form of inequalities. Technical constraints are set by the state of technology and the availability of factors of production. The number of technical constraints in a linear programming problem is equal to the number of factors involved it.

(3) Non-Negativity Constraints

This express the level of production of the commodity cannot be negative, ie it is either positive or zero.

(4) Feasible Solutions

After knowing the constraints, feasible solutions of the problem for a consumer, a particular, a firm or an economy can be ascertained. Feasible solutions are those which meet or satisfy the constraints of the problem and therefore it is possible to attain them.

(5) Optimum Solution

The best of all feasible solutions is the optimum solution. In other words, of all the feasible solutions, the solution which maximizes or minimizes the objective function is the optimum solution. For instance, if the objective function is to maximize profits from the production of two goods, then the optimum solution will be that combination of two products that will maximizes the profits for the firm. Similarly, if the objective function is to minimize cost by the choice of a process or combination of processes, then the process or a combination of processes which actually minimizes the cost will represent the optimum solution. It is worthwhile to repeat that optimum solution must lie within the region of feasible solutions.

2.13 Assumptions of LPP

The LPP are solved on the basis of some assumptions which follow from the nature of the problem.

(a) Linearity

The objective function to be optimized and the constraints involve only linear relations. They should be linear in their variables. If they are not, alternative technique to solve the problem has to be found. Linearity implies proportionality between activity levels and resources. Constraints are rules governing the process.

(b) Non-negativity

The decision variable should necessarily be non-negative.
(c) **Additive and divisibility**

Resources and activities must be additive and divisible.

(d) **Alternatives**

There should be alternative choice of action with a well defined objective function to be maximized or minimized.

(e) **Finiteness**

Activities, resources, constraints should be finite and known.

(f) **Certainty**

Prices and various coefficients should be known with certainty.

2.14 **Application of linear programming**

There is a wide variety of problem to which linear programming methods have been successfully applied.

- **Diet problems**

  To determine the minimum requirements of nutrients subjects to availability of foods and their prices.

- **Transportation problem**

  To decide the routes, number of units, the choice of factories, so that the cost of operation is the minimum.

- **Manufacturing problems**

  To find the number of items of each type that should be made so as to maximize the profits.

- **Production problems**

  Subject to the sales fluctuations. To decide the production schedule to satisfy demand and minimize cost in the face of fluctuating rates and storage expenses.

- **Assembling problems**

  To have, the best combination of basic components to produce goods according to certain specifications.

- **Purchasing problems**

  To have the least cost objective in, say, the processing of goods purchased from outside and varying in quantity, quality and prices.
• **Job assigning problem**

To assign jobs to workers for maximum effectiveness and optimum results subject to restrictions of wages and other costs.

### 2.15 Limitations of LPP

The computations required in complex problems may be enormous. The assumption of divisibility of resources may often be not true. Linearity of the objective function and constraints may not be a valid assumption. In practice work there can be several objectives, not just a single objective as assumed in LP.

### 2.16 Formulation of Linear Programming

The formulation has to be done in an appropriate form. We should have,

1. An objective function to be maximized or minimized. It will have \(n\) decision variables \(x_1, x_2, \ldots, x_n\) and is written in the form.

   \[
   \text{Max (Z) or Min(c)} = C_1 x_1 + C_2 x_2 + \ldots + C_n x_n
   \]

   Where each \(C_j\) is a constant which stands for per unit contribution of profit (in the maximization case) or cost (in the minimization case to each \(X_j\)).

2. The constraints in the form of linear inequalities.

   \[
   \begin{align*}
   a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n & \leq b_1 \\
   a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n & \leq b_2 \\
   \vdots & \vdots \\
   a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n & \leq b_n
   \end{align*}
   \]

   Briefly written

   \[
   \sum a_{ij} x_j \leq b_i \quad I = 1, 2, \ldots, n
   \]

   Where \(b_i\) stands for the \(i^{th}\) requirement or constraint.

   The non-negativity constraints are

   \[
   x_1, x_2, \ldots, x_n \geq 0
   \]

In matrix notation, we write

Max or Min \(Z = CX\)

Subject to constraints \(AX \leq b\) or \(AX \geq b\) and the non-negativity conditions \(x \geq 0\).
Where \( C = [ c_1, c_2, \ldots c_n ] \) Here

\[
\begin{bmatrix}
  a_{11} & a_{12} & \cdots & a_{1n} \\
  a_{21} & a_{22} & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{m1} & a_{m2} & \cdots & a_{mn}
\end{bmatrix}
\begin{bmatrix}
  x_1 \\
  x_2 \\
  \vdots \\
  x_n
\end{bmatrix}
\begin{bmatrix}
  b_1 \\
  b_2 \\
  \vdots \\
  b_m
\end{bmatrix}
\]

**Examples:** A firm can produce a good either by (1) a labour intensive technique, using 8 units of labour and 1 unit of capital or (2) a capital intensive technique using 1 unit of labour and 2 units of capital. The firm can arrange up to 200 units of labour and 100 units of capital. It can sell the good at a constant net price \( P \), i.e., \( P \) is obtained after subtracting costs. Obviously, we have simplified the problem because in this ‘P’ become profit per unit. Let \( P = 1 \).

Let \( x_1 \) and \( x_2 \) be the quantities of the goods produced by the processes 1 and 2 respectively.

To maximize the profit \( P \cdot x_1 + P \cdot x_2 \), we write the objective function.

\[ \Pi = x_1 + x_2 \] (since \( P = 1 \)). The problem becomes

\[ \text{Max } \pi = x_1 + x_2 \]

Subject to: The labour constraint \( 8 \cdot x_1 + x_2 \leq 200 \)

The capital constraint \( x_1 + x_2 \leq 100 \)

And the non-negativity conditions \( x_1 \geq 0, x_2 \geq 0 \)

This is a problem in linear programming.

**Example:** Two foods \( F_1, F_2 \) are available at the prices of Rs. 1 and Rs. 2 per unit respectively. \( N_1, N_2, N_3 \) are essential for an individual. The table gives these minimum requirements and nutrients available from one unit of each of \( F_1, F_2 \). The question is of minimizing cost \( C \), while satisfying these requirements.

<table>
<thead>
<tr>
<th>Nutrients</th>
<th>Minimum requirements</th>
<th>One units of ( F_1 )</th>
<th>One units of ( F_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( N_1 )</td>
<td>17</td>
<td>9</td>
<td>2</td>
</tr>
<tr>
<td>( N_2 )</td>
<td>19</td>
<td>3</td>
<td>4</td>
</tr>
<tr>
<td>( N_3 )</td>
<td>15</td>
<td>2</td>
<td>5</td>
</tr>
</tbody>
</table>
Total Cost (TC)  \( c = P_1 x_1 + P_2 x_2 \)  
\( (x_1, x_2 \) quantities of F_1, F_2)  

Where \( P_1 = 1, P = 2 \)  

We therefore have to Minimize  

\[ C = x_1 + 2x_2 \]

Subject to the minimum nutrient requirement constraints,  

\[ 9x_1 + x_2 \geq 17 \]
\[ 3x_1 + 4x_2 \geq 19 \]
\[ 2x_1 + 5x_2 \geq 15 \]

Non- negativity conditions  

\[ x_1 \geq 0, x_2 \geq 0. \]

**Graphical Solution**

If the LPP consist of only two decision variable. We can apply the graphical method of solving the problem. It consists of seven steps, they are  

1. Formulate the problem in to LPP.  
2. Each inequality in the constraint may be treated as equality.  
3. Draw the straight line corresponding to equation obtained steps (2) so there will be as many straight lines, as there are equations.  
4. Identify the feasible region. This is the region which satisfies all the constraints in the problem.  
5. The feasible region is a many sided figures. The corner point of the figure is to be located and they are coordinate to be measures.  
6. Calculate the value of the objective function at each corner point.  
7. The solution is given by the coordinate of the corner point which optimizes the objective function.

**Example**: Solve the following LPP graphically.  

Maximize \( Z = 3x_1 + 4x_2 \)  

Subject to the constraints  

\[ 4x_1 + 2x_2 \leq 80 \]
\[ 2x_1 + 5x_2 \leq 180 \]
\[ x_1, x_2 \geq 0 \]

Treating the constraints are equal, we get

\[ 4x_1 + 2x_2 = 80 \] .......................... (1)
\[ 2x_1 + 5x_2 = 180 \] .......................... (2)
\[ x_1 = 0 \] .......................... (3)
\[ x_2 = 0 \] .......................... (4)

In equation (1), putting \( x_1 = 0 \), we get

\[ 0x_1 + 2x_2 = 80 \]
\[ x_2 = 80/2 \]
\[ = 40 \]

When \( x_2 = 0 \)

\[ 4x_1 + 0x_2 = 80 \]
\[ x_1 = 80/4 \]
\[ = 20 \]

So \((0, 40)\) and \((20, 0)\) are the two points on the straight line given by equation (1)

Similarly in the equation (2), we get

\[ x_1 = 0 \]
\[ 0x_1 + 5x_2 = 180 \]
\[ x_2 = 180/5 \]
\[ = 36 \]
\[ x_2 = 0 \]
\[ 2x_1 + 0x_2 = 180 \]
\[ x_1 = 180/2 \]
\[ = 90 \]

Therefore \((0, 36)\) and \((90, 0)\) are the two points on the straight line represent by the equation (2).
The equation (3) and (4) are representing the $x_1$ and $x_2$ axis respectively.

The feasible region is the (shaded area) shaded portion, it has four corner points, say OABC

The coordinate of O = (0, 0)  
A = (20, 0)  
C = (0, 36) and B can be obtained by solving the equations for the lines passing through that point.

The equations are (1) and (2)

\[ 4x_1 + 2x_2 = 80 \]  \hspace{2cm} (1)
\[ 2x_1 + 5x_2 = 180 \]  \hspace{2cm} (2)

Then (2) – (1) \[ 0 + 4x_2 = 140 \]
\[ x_2 = 140/4 \]
\[ = 35 \]

Substituting the value of $x_2$ in (1), we get

\[ x_1 = 40 - 35 / 2 \]
\[ = 5/2 \]
\[ = 2.5 \]

So the coordinate b are \((x_1 = 2.5, x_2 = 35)\)

Evaluate the objective function of the corner points is given below.
<table>
<thead>
<tr>
<th>Point</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$Z$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>A</td>
<td>20</td>
<td>0</td>
<td>60</td>
</tr>
<tr>
<td>B</td>
<td>2.5</td>
<td>35</td>
<td>147.5</td>
</tr>
<tr>
<td>C</td>
<td>0</td>
<td>36</td>
<td>144</td>
</tr>
</tbody>
</table>

B gives the maximum value of $Z$, so the solution is $x_1 = 2.5$ and $x_2 = 35$

Maximum value if $Z = 147.5$

**Example:** Solve the following LPP graphically.

Minimize $C = 6x_1 + 11x_2$

Subject to the constraints

\[2x_1 + x_2 \geq 104\]
\[x_1 + 2x_2 \geq 76\]
\[x_1, x_2 \geq 0\]

Treating the constraints are equal, we get

\[2x_1 + x_2 = 104 \quad \text{............... (1)}\]
\[x_1 + 2x_2 = 76\quad \text{............... (2)}\]
\[x_1 = 0\quad \text{............... (3)}\]
\[x_2 = 0\quad \text{............... (4)}\]

In equation (1), putting $x_1 = 0$, we get

\[0 + x_2 = 104\]
\[x_2 = 104\]

When $x_2 = 0$

\[2x_1 = 104\]
\[2x_1 = 104/2\]
\[= 52\]
So \((0, 104)\) and \((52, 0)\) are the two points in the straight line given by equation (1)

Similarly in the equation (2), we get
\[
\begin{align*}
x_1 &= 0 \\
0x_1 + 2x_2 &= 76 \\
2x_2 &= 76 \\
x_2 &= 76/2 \\
&= 38 \\
x_1 + 0x_2 &= 76 \\
x_1 &= 76
\end{align*}
\]

Therefore \((0, 38)\) and \((76, 0)\) are the two points on the straight line represented by the equation (2). The equation (3) and (4) are representing the \(x_1\) and \(x_2\) axis respectively.

The feasible region is the (shaded area) shaded portion, it has three corner points, say PNM

The coordinate of
\[
\begin{align*}
M &= (52, 0) \\
P &= (0, 38) \\
N &\text{ can be obtained by solving the equations for the lines passing through that point.}
\end{align*}
\]

The equations are (1) and (2)
\[
\begin{align*}
2x_1 + x_2 &\geq 104 \quad \ldots \quad (1) \\
x_1 + 2x_2 &\geq 76 \quad \ldots \quad (2)
\end{align*}
\]

Taking the second constraint and multiplied by 2 throughout the equation
\[2x_1 + x_2 = 104 \quad \ldots \ldots \quad (1)\]
\[2x_1 + 4x_2 = 152 \quad \ldots \ldots \quad (2)\]

Then \((2) – (1)\)
\[0 + 3x_2 = 48\]
\[x_2 = \frac{48}{3}\]
\[= 16\]

Substituting the value of \(x_2\) in \((1)\), we get
\[2x_1 + 16 = 104\]
\[= 104 – 16/2\]
\[= 88/2\]
\[= 44\]

So the coordinate b are \((x_1 = 88, x_2 = 16)\)

Evaluate the objective function of the corner points is given below.

<table>
<thead>
<tr>
<th>Point</th>
<th>(x_1)</th>
<th>(x_2)</th>
<th>(Z)</th>
</tr>
</thead>
<tbody>
<tr>
<td>P</td>
<td>(0, 104)</td>
<td>1144</td>
<td>6×0 + 11×104=1144</td>
</tr>
<tr>
<td>N</td>
<td>(46, 17)</td>
<td>440</td>
<td>6×44+14×16=440</td>
</tr>
<tr>
<td>M</td>
<td>(75, 0)</td>
<td>450</td>
<td>6×75+11×0=450</td>
</tr>
</tbody>
</table>

N gives the minimum value of \(C\), so the solution is \(x_1 = 46\) and \(x_2 = 17\)

Minimum value of \(C = 440\)

**Exercise:** A baker has 150 kilograms of flour, 22 kilos of sugar, and 27.5 kilos of butter with which to make two types of cake. Suppose that making one dozen A cakes requires 3 kilos of flour, kilo of butter, whereas making one dozen B cakes requires 6 kilos of flour, 0.5 kilo of sugar, and 1 kilo of butter. Suppose that the profit from one dozen A cakes is 20 and from one dozen B cakes is 30. How many dozen a cakes \((x_1)\) and how many dozen B cakes \((x_2)\) will maximize the baker’s profit?

**Solution:**

An output of \(x_1\) dozen and \(x_2\) dozen B cakes would need \(3x_1 + 6x_2\) kilos of flour. Because there are only 150 kilos of flour, the inequality.

\[3x_1 + 6x_2 \leq 150 \quad \text{(flour constraint)\ldots\ldots (1)}\]

Similarly, for sugar,

\[x_1 + 0.5x_2 \leq 22 \quad \text{(sugar constraint)\ldots\ldots(2)}\]

and for butter,
\[ x_1 + x_2 \leq 27.5 \text{ (butter constraint)} \] \tag{3}

Of course, \( x_1 \geq 0 \) and \( x_2 \geq 0 \). The profit obtained from producing \( x_1 \) dozen A cakes and \( x_2 \) dozen B cakes is

\[ Z = 20x_1 + 30x_2 \] \tag{4}

In short, the problem is to

\[
\begin{align*}
\text{Max } Z &= 20x_1 + 30x_2 \\
\text{s.t } &3x_1 + 6x_2 \leq 150 \\
&x_1 + 0.5x_2 \leq 22 \\
&x_1 + x_2 \leq 27.5 \\
&x_1 \geq 0 \\
&x_2 \geq 0
\end{align*}
\]

**Exercise:** Solve Graphically,

\[
\begin{align*}
\text{Max } Z &= 3x_1 + 4x_2 \\
\text{Sub to } &3x_1 + 2x_2 \leq 6 \\
&x_1 + 4x_2 \leq 4 \\
&x_1 \geq 0, x_2 \geq 0 \\
\text{Min } C &= 10x_1 + 27x_2 \\
\text{Sub to } &x_1 + 3x_2 \geq 11 \\
&2x_1 + 5x_2 \geq 20 \\
&x_1 \geq 0, x_2 \geq 0 \\
\text{Max } Z &= 2x + 7y \\
\text{Sub to } &4x + 5y \leq 20 \\
&3x + 7y \leq 21 \\
&x_1 \geq 0, x_2 \geq 0
\end{align*}
\]

**FURTHER READINGS**

1. R.G.D Allen - Mathematical Economics.
2. Taro Yamane - Mathematics for Economics.
III.1. Market Equilibrium

The equilibrium price of a good is that price where the supply of the good equals the demand. Geometrically, this is the price where the demand and the supply curves cross. If we let \( D(p) \) be the market demand curve and \( S(p) \) the market supply curve, the equilibrium price is the price \( p^* \) that solves the equation.

\[
D(p) = S(p)
\]

The solution to this equation, \( p^* \), is the price where market demand equals market supply. When market is in equilibrium, then there is no excess demand and supply.

Assuming that both demand and supply curves are linear, demand – supply model can be stated in the form of the following equation.

\[
q_D = a - bp \quad \quad (1)
\]
\[
q_S = c + dp \quad \quad (2)
\]
\[
q_D = q_S \quad \quad (3)
\]

Where \( q_d \) and \( q_s \) are quantities demanded and supplied respectively, \( a \) and \( c \) are intercept coefficients of demand and supply curves respectively, \( b \) and \( d \) are the coefficients that measures the slop of these curves, equation (3) is the equilibrium condition.

Thus in equilibrium

\[
a - bp = c + dp
\]

\[
a - c = dp + bp = p(d + b)
\]

Dividing both sides by \( d + b \) we have

\[
\frac{a - c}{d + b} = p
\]

Or equilibrium price

\[
P = \frac{a - c}{d + b} \quad \quad (4)
\]

Substituting (4) into (1) we have equilibrium quantity
Equation (4) and (5) describe the qualitative results of the model. If the values of the parameters $a$, $b$, $c$ and $d$ are given we can obtain the equilibrium price and quantity by substituting the values of these parameters in the qualitative results of equation (4) and (5).

**Numerical Example:**

Suppose the following demand and supply functions of a commodity are given which is being produced under perfect competition. Find out the equilibrium price and quantity.

\[ q^d = 750 - 25p \]
\[ q^s = 300 + 20p \]

**Solution:** There are two alternatives ways of solving for equilibrium price and quantity.

First we can find out the equilibrium price and quantity by using the equilibrium condition, namely $q^d = q^s$.

Second, we can obtain equilibrium price and quantity by using the qualitative results of the demand and supply model.

\[ p = \frac{a-c}{d+b} \]
\[ q = \frac{ad+bc}{b+d} \]

1. Since in equilibrium $q^d = q^s$

\[ 750 - 25p = 300 + 20p \]
\[ 45p = 750 - 300 \]
\[ P = \frac{450}{45} = 10 \]

Now substitute the value of $p$ in the demand function

\[ q^d = 750 - 25p \]
\[ = 750 - 25 \times 10 \]
\[ = 500 \]

**Alternative Method:**

\[ P = \frac{a-c}{d+b} \]
Where $a = 750$, $b = 25$, $c = 300$, $d = 20$

$$P = \frac{750 - 300}{20 + 25} = \frac{450}{45} = 10$$

$$q^d = \frac{ad + bc}{b + d}$$

$$= \frac{750 \times 20 + 25 \times 300}{20 + 25}$$

$$= \frac{15000 \times 750}{45}$$

$$= \frac{22500}{45} = 500$$


In the perfect competitive market the firms are in equilibrium when they maximize their profits ($\Pi$). The profit is the difference between the total cost and total revenue, i.e, $\Pi = TR - TC$.

The conditions for equilibrium are

1. $MC = MR$

2. Slope of $MC >$ slope of $MR$

Derivation of the equilibrium of the firm

The firms aims at the maximization of its profit

$$\Pi = TR - TC$$

Where

$\Pi = \text{Profit}$

$TR = \text{Total Revenue}$

$TC = \text{Total cost}$

Clearly $TR = \int_1^n (X)$ and $TC = \int_2^n (X)$, given the price $p$.

(a) The first-order condition for the maximization of a function is that its first derivative (with respect to $X$ in our case) be equal to zero. Differentiating the total-profit function and equating to zero we obtain

$$\frac{d\Pi}{dX} = \frac{d(TR)}{dX} - \frac{d(TC)}{dX} = 0$$

Or

$$\frac{d(TR)}{dX} = \frac{d(TC)}{dX}$$
The term \( \frac{d(TR)}{dX} \) is the slope of the total revenue curve, that is, the marginal revenue. The term \( \frac{d(T)}{dX} \) is the slope of the total cost curve, or the marginal cost. Thus the first-order condition for profit maximization is

\[ MR = MC \]

Given that \( MR > 0 \), MR must also be positive at equilibrium. Since \( MR = P \) the first-order condition may be written as \( MC = P \).

(b) The second-order condition for a maximum requires that the second derivative of the function be negative (implying that after its highest point the curve turns downwards). The second derivative of the total-profit function is

\[
\frac{d^2(P)}{dX^2} = \frac{d^2(TR)}{dX^2} - \frac{d^2(TC)}{dX^2}
\]

This must be negative if the function has been maximized, that is

\[
\frac{d^2(TR)}{dX^2} - \frac{d^2(TC)}{dX^2} < 0
\]

which yields the condition?

\[
\frac{d^2(TR)}{dX^2} < \frac{d^2(TC)}{dX^2}
\]

But \( \frac{d^2TR}{dX^2} \) is the slope of MR curve and \( \frac{d^2TC}{dX^2} \) is the slope of the MC curve. Hence the second-order condition may verbally be written as follows

\((\text{slope of MR}) < (\text{slope of MC})\)

Thus the MC must have a steeper slope than the MR curve or the MC must cut the MR curve from below. In pure competition the slope of the MR curve is zero, hence the second-order condition is simplified as follows.

\[
0 < \frac{d^2(TC)}{dX^2}
\]

Which reads: the MC curve must have a positive slope, or the MC must be rising.

**Numerical Example:** A perfectly competitive market faces \( P = \text{Rs. 4} \) and \( TC = X^3 - 7X^2 + 12X + 5 \).

Find the best level of output of the firm. Also find the profit of the firm at this level of output.

First condition requires, \( MR = MC \)

\[ TR = PX = 4X, \text{ as } P = 4 \]
MR = \frac{dTR}{dX} = 4, which is also equal to price. So MR = 4 = P

MC = \frac{dTC}{dX} = 3X^2 - 14X + 12

Setting MR = MC and solving for X to find the critical values

4 = 3X^2 - 14X + 12
3X^2 - 14X + 12 - 8 = 0
3X^2 - 14X + 8 = 0

By factorization we have the values as

(3X - 2) and (X - 4)

So 3X = 2, X = \frac{2}{3} and X = 4

This means that at the equilibrium point MR = MC, X = \frac{2}{3} and X = 4

The second condition requires that MC must be rising at this point of intersection. In other words, the slope of the MC curve should be positive at the point where MC = MR. The equation for the slope of the MC curve is to find its derivatives.

\frac{dMC}{dX} = 6X - 14

Then substitute the two critical values X = \frac{2}{3} and X = 4 in the above equation to find out the point which maximize the profit.

When X = \frac{2}{3}, 6X - 14 would be 6\times\frac{2}{3} - 14 = -10. It is not the profit maximizing output.

When X = 4, 6X - 14 would be 6\times4 - 14 = 10.

Here the profit is maximized when the output is equal to 10 units.

Then we have to find the maximum profit. The maximum profit is obtained when the output is at 10 units. So substitute the value, i.e, X=4 in the profit function.

Then, \Pi = TR - TC

\Pi = 4X - (X^3 - 7X^2 + 12X + 5)
= 4X - X^3 + 7X^2 - 12X - 5
= -X^3 - 7X^2 - 8X - 5
= -64 + 112 - 32 - 5
= 11
The firm maximizes its profit at the output level of 4 units and at this level its maximum profit is Rs.11.

III.3. Equilibrium in the Monopoly

Monopoly is a market structure in which there is a single seller, there are no close substitutes for the commodity it produces and there are barriers to entry.

A. Short-run Equilibrium

The monopolist maximizes his short-run profit if the following two conditions are fulfilled:

1. The MC is equal to the MR, i.e., MC = MR
2. The slope of the MC is greater than the slope of the MR at the point of intersection.

Mathematical derivation

The given demand function is $X = g(P)$

Which may be solved for $P$, $P = \int_1 (X)$

The given cost function is $C = \int_2 (X)$

The monopolist aims at the maximization of his profit

$$\Pi = TR - TC$$

(a) The first-order condition for maximum profit $\Pi$

$$\frac{d\Pi}{dX} = 0$$

$$\frac{d(\Pi)}{dX} = \frac{d(TR)}{dX} - \frac{d(TC)}{dX} = 0$$

Or

$$\frac{d(TR)}{dX} = \frac{d(TC)}{dX}$$

That is $MR = MC$

(b) The second-order condition for maximum profit

$$\frac{d^2\Pi}{dX^2} < 0$$

$$\frac{d^2\Pi}{dX^2} = \frac{d^2(TR)}{dX^2} - \frac{d^2(TC)}{dX^2} < 0$$
Or

\[ \frac{d^2(TR)}{dx^2} < \frac{d^2(TC)}{dx^2} \]

That is

[slope of MR] < [slope of MC]

**Numerical Example:**

Given the demand curve of the monopolist

\[ X = 50 - 0.5p \]

Which may be solved for \( P \):

\[ P = 100 - 2X \]

Given the cost function of the monopolist

\[ TC = 50 + 40X \]

The goal of the monopolist is to maximize profit

\[ \Pi = TR - TC \]

(i) First find the MR

\[ TR = XP = X (100 - 2X) \]

\[ TR = 100X - 2X^2 \]

\[ MR = \frac{dTR}{dx} = 100 - 4X \]

(ii) Next find the MC

\[ TC = 50 + 40X \]

\[ MC = \frac{dTC}{dx} = 40 \]

(iii) Equate MR and MC

\[ MR = MC \]

\[ 100 - 4X = 40 \]

\[ X = 15 \]
(iv) The monopolist’s price is found by substituting $X = 15$ into the demand-price equation

$$P = 100 - 2X = 70$$

(v) The profit is

$$\Pi = TR - TC = 1050 - 650 = 400$$

This profit is the maximum possible, since the second-order condition is satisfied:

(a) From

$$\frac{dT C}{dX} = 40$$

We have

$$\frac{d^2(TC)}{dX^2} = 0$$

(b) From

$$\frac{dTR}{dX} = 100 - 4X$$

we have $\frac{d^2(TR)}{dX^2} = -4$

Clearly $-4 < 0$

**Alternative Method**

The same problem can be worked out by another method.

After finding $TR$ and $TC$, compute the profit function $\Pi$.

$$\Pi = TR - TC = 100X - 2X^2 - (50 + 40X)$$

$$= 100X - 2X^2 - 50 + 40X$$

$$= 60X - 2X^2 - 50$$

$$\Pi = -2X^2 + 60X - 50$$

As per the optimization rule, we can optimize the function. At first find the first order derivative and equate it with zero and find the critical value.

$$\Pi' = -4X + 60 = 0$$

$$-4X + 60 = 0, -4X = -60$$

$$X = 15$$

The second order condition for the maximisation must be less than zero.
\[ \Pi'' = -4 < 0 \]
\[ \Pi''(X=15) = -4 < 0 \]

Here the conditions for the maximisation have been satisfied. So the function is maximized at \( X = 15 \).

To find the price, \( P \), substitute \( X = 15 \) in the demand function \( P = 100 - 2X \)

i.e, \( P = 100 - 2(15) = 100 - 30 = 70 \)

The monopolist maximize his profit when the quantity \( X = 15 \) and the price, \( P = 70 \).

The maximum profit of the monopolist,

\[ \Pi = -2X^2 + 60X - 50 \]

Substitute \( \Pi = -2(15^2) + 60(15) - 50 = 400 \).

**B. Long-run Equilibrium**

As you know that, in the long run the monopolist has the time to expand his plant, or to use his existing plant at any level which will maximize his profit.

Mathematical derivation of the equilibrium of the multi-plant monopolist

Given the market demand

\[ P = f(X_1 + X_2) \]

And the cost structure of the plants

\[ TC_1 = f_1(X_1) \]
\[ TC_2 = f_2(X_2) \]

The monopolist aims at the allocation of his production between plant A and plant B so as to maximize his profit

\[ \Pi = TR - TC_1 - TC_2 \]

The first-order condition for maximum profit requires

\[ \frac{d\Pi}{dX_2} = 0 \text{ and } \frac{d\Pi}{dX_1} = 0 \]

(a)

\[ \frac{d\Pi}{dX_1} = \frac{dTR}{dX_1} - \frac{dTC_1}{dX_1} = 0 \]
Or

$$\frac{dTR}{dx} - \frac{dTC}{dx} = 0$$

i.e., \( MR_1 = MC_1 \)

(a) \[ \frac{d\Pi}{dx} = \frac{dTR}{dx} - \frac{dTC}{dx} = 0 \]

Or

$$\frac{dTR}{dx} - \frac{dTC}{dx} = 0$$

i.e., \( MR_2 = MC_2 \)

But \( MR_1 = MR_2 = MR \) (given that each unit of the homogeneous output will be sold at the same price \( P \) and will yield the same marginal revenue, irrespective in which plant the unit has been produced)

Therefore

\[ MR = \frac{dTC}{dx} = MC_1 \] and \[ MR = \frac{dTC}{dx} = MC_2 \]

So that \( MR = MC_1 = MC_2 \)

The second-order condition for maximum profit requires

That is, the MC in each plant must be increasing more rapidly than the (common) MR of the output as a whole.

**Numerical Example:**

The monopolist’s demand curve is

\[ X = 200 - 2p, \text{ or } p = 100 - 0.5X \]

The costs of the two plants are

\[ C_1 = 10X_1 \text{ and } C_2 = 0.25X_2^2 \]

The goal of the monopolist is to maximize profit

\[ \Pi = TR - TC_1 - TC_2 \]

(1) \[ TR = Xp = X(100 - 0.5X) \]

\[ TR = 100X - 0.5X^2 \]
\[ \text{MR} = \frac{dTR}{dX} = 100 - X = 100 - (X_1 + X_2) \]
\[ C_1 = 10X_1 \]
\[ MC_1 = \frac{dTC_1}{dX_1} = 10 \]

And
\[ C_2 = 0.25X_2^2 \]
\[ MC_2 = \frac{dTC_2}{dX_2} = 0.5X_2 \]

(3) Equating each MC to the common MR

\[ 100 - X_1 - X_2 = 10 \]
\[ 100 - X_1 - X_2 = 0.5X_2 \]

Solving for \( X_1 \) and \( X_2 \) we find

\[ X_1 = 70 \text{ and } X_2 = 20 \]

So that the total \( X \) is 90 units. This total output will be sold at price \( P \) defined by

\[ P = 100 - 0.5X = 55 \]

The monopolist’s profit is

\[ \Pi = TR - TC_1 - TC_2 \]
\[ = 4950 - 10(20) - 0.25(4900) \]
\[ \Pi = 3525 \]

This is the maximum profit since the second-order condition is fulfilled.

**III.4. Discriminating Monopoly**

Price discrimination exists when the same product is sold at different prices to different buyers. The cost of production is either the same, or it differs but not as much as the difference in the charged prices. The product is basically the same, but it may have slight differences. Here we consider the typical case of an identical product, produced at the same cost, which is sold at different prices, depending on the preferences of the buyers, their income, their location and the ease of availability of substitutes. These factors give rise to demand curves with different elasticities in the various sectors of the market of a firm. There also charges different prices for the same product at different time periods.
Mathematical derivation of the equilibrium position of the price-discriminating monopolist

Given the total demand of the monopolist

\[ P = f(X) \]

Assume that the demand curves of the segmented markets are

\[ P_1 = f_1(X_1) \quad \text{and} \quad P_2 = f_2(X_2) \]

The cost of the firm is

\[ C = f(X) = f(X_1 + X_2) \]

The firm aims at the maximisation of its profit

\[ \Pi = TR_1 - TR_2 - C \]

The first-order condition for profit maximisation requires

\[ \frac{d\Pi}{dX_1} = 0 \quad \text{and} \quad \frac{d\Pi}{dX_2} = 0 \]

(a) \[ \frac{d\Pi}{dX_1} = \frac{dTR_1}{dX_1} - \frac{dTC}{dX_1} = 0 \quad \text{and} \quad \frac{d\Pi}{dX_2} = \frac{dTR_2}{dX_2} - \frac{dTC}{dX_2} = 0 \]

(b) \[ \frac{dTR_1}{dX_1} - \frac{dTC}{dX_1} \quad \text{or} \quad MR_1 = MC_1; \quad \text{and} \quad \frac{dTR_2}{dX_2} - \frac{dTC}{dX_2} \quad \text{or} \quad MR_2 = MC_2 \]

But

\[ MC_1 = MC_2 = MC = \frac{dTC}{dX} \]

Therefore

\[ MC = MR_1 = MR_2 \]

The second-order condition for profit maximisation requires

\[ \frac{d^2TR_1}{dX_1^2} < \frac{d^2TC}{dX_1^2} \quad \text{and} \quad \frac{d^2TR_2}{dX_2^2} < \frac{d^2TC}{dX_2^2} \]

That is, the MR in each market must be increasing less rapidly than the MC for the output as a whole.

**Numerical Example:**

The total demand function is

\[ X = 50 - 0.5 P \quad \text{or} \quad P = 100 - 2X \]
Assuming that the demand function of segmented markets are

\[ X_1 = 32 - 0.4P_1 \text{ or } P_1 = 80 - 2.5X_1 \]
\[ X_2 = 18 - 0.1P_2 \text{ or } P_2 = 180 - 10X_2 \]

That is \( X_1 + X_2 = X \)

The Cost function is

\[ C = 50 + 40X = 50 + 40(X_1 + X_2) \]

The firm aims at the maximisation of its profit

i.e, \( \Pi = TR_1 - TR_2 - TC \)

**Solution**

(1) \( TR_1 = X_1P_1 = X_1(80 - 2.5X_1) = 80X_1 - 2.5X_1^2 \)

\( MR_1 = \frac{dT_1}{dX_1} = 80 - 5X_1 \)

(2) \( TR_2 = X_2P_2 = X_2(180 - 10X_2) = 180X_2 - 10X_2^2 \)

\( MR_2 = \frac{dT_2}{dX_2} = 180 - 20X_2 \)

(3) \( MC = \frac{dTC}{dX_1} = \frac{dTC}{dX_2} = \frac{dTC}{dX} = 40 \)

Setting the MR in each market equal to the common MC we obtain

\[ 80 - 5X_1 = 40 \]
\[ 180 - 20X_2 = 40 \]

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\[ X_1 = 8 \]
\[ X_2 = 7 \]

Here total \( X = 15 \)

The prices are

\[ P_1 = 80 - 2.5X_1 = 60 \]
\[ P_2 = 180 - 10X_2 = 110 \]
The profit is
\[ \Pi = TR_1 \cdot TR_2 - TC = 500 \]

The elasticities are
\[ e_1 = \frac{dX_1}{dP_1} \cdot \frac{P_1}{X_1} = (0.4) \frac{60}{8} = 3 \]
\[ e_2 = \frac{dX_2}{dP_2} \cdot \frac{P_2}{X_2} = (0.1) \frac{110}{7} = 1.57 \]

Thus \( e_1 > e_2 \) and \( P_1 < P_2 \)

Comparing the above results with those for the example of the simple monopolist we observe that \( X \) is the same in both cases but the \( \Pi \) of the discriminating monopolist is larger.

### III.5. Price Discrimination and the Price Elasticity of Demand

As we know that, the relationship between MR and price elasticity \( e \) is

\[ MR = P \left( 1 - \frac{1}{e} \right) \]

Proof

\[ MR = \frac{dT}{dX} = P + X \frac{dP}{dX} \]

The price elasticity of demand is defined as

\[ e_p = -\frac{dP}{dX} \cdot \frac{X}{P} \]

Inverting this relation we obtain

\[ \frac{1}{e} = \frac{dP}{dX} \cdot \frac{X}{P} \]

Solving for \( dP/dX \) then we have

\[ \frac{dP}{dX} = -\frac{1}{e} \cdot \frac{X}{P} \]

Substituting in the expression of the MR we get

\[ MR = P + X \left( -\frac{1}{e} \cdot \frac{F}{X} \right) \]

Or

\[ MR = P \left( 1 - \frac{1}{e} \right) \]
So, in the case of price discrimination we have

\[ MR_1 = P_1 \left(1 - \frac{1}{e_1}\right) \]

\[ MR_2 = P_2 \left(1 - \frac{1}{e_2}\right) \]

And

\[ MR_1 = MR_2 \]

Therefore

\[ P_1 \left(1 - \frac{1}{e_1}\right) = P_2 \left(1 - \frac{1}{e_2}\right) \]

Or

\[ \frac{P_1}{P_2} = \frac{\left(1 - \frac{1}{e_1}\right)}{\left(1 - \frac{1}{e_2}\right)} \]

Where \( e_1 = \) elasticity of \( D_1 \)

\( e_2 = \) elasticity of \( D_2 \)

If \( e_1 = e_2 \) the ratio of prices is equal to unity:

\[ \frac{P_1}{P_2} = 1 \]

That is, \( P_1 = P_2 \). This means that when elasticities are the same price discrimination in not profitable. The monopolist will charge a uniform price for his product.

If price elasticities differ price will be higher in the market whose demand is less elastic. This is obvious from the equality of MR’s

\[ P_1 \left(1 - \frac{1}{e_1}\right) = P_2 \left(1 - \frac{1}{e_2}\right) \]

If \( |e_2| > |e_1| \), the

\( \left(1 - \frac{1}{e_1}\right) > \left(1 - \frac{1}{e_2}\right) \)

Thus for the equality of MR’s to be fulfilled

\[ P_1 < P_2 \]

That is, the market with the higher elasticity will have the lower price.
References & Further Readings


MODULE IV

NATURE AND SCOPE OF ECONOMETRICS

Econometrics: Meaning, Scope, and Limitations - Methodology of econometrics – Types of data: Time series, Cross section and panel data.

4.1 Nature and Scope of Econometrics

Econometrics means economic measurement. It deals with the measurement of economic relationships. The term econometrics is formed from two words – economy and measure. It was Ragnar Frisch (1936) who coined the term Econometrics. The term econometrics was first used by Pawel Clompa in 1910. But the credit of coining the term econometrics should be given to Ragnar Frisch (1936), one of the founders of the Econometric Society. He was the person who established the subject in the sense in which it is known today. Econometrics can be defined generally as “the application of mathematics and statistical methods to the analysis of economic data”.

Econometrics is a combination of economic theory, mathematical economics and statistics. It may be considered as the integration of economics, mathematics and statistics for the purpose of providing numerical value for economic relationships and for verifying economic theories.

4.2 Definitions

1. Econometrics may be defined as the quantitative analysis of actual economic phenomenon based on the concurrent development of theory and observation, related by appropriate methods of inference. (P.A. Samuelson, T.C.Koopman, J.R.N Stone)

2. Econometrics is concerned with the empirical determination of economic laws (H.Theil)

3. Econometrics may be defined as the social science in which the tools of economic theory, mathematics and statistical inference are applied to the analysis of economic phenomena (Arthur S.Goldberg)

4. Econometrics consists of the application of mathematical statistics to economic data to lend empirical support to the model constructed on mathematical economics and to obtain numerical results. (Gerhard. Tinter)

5. Every application of mathematics or of statistical methods to the study of economic phenomena (Malinvaud 1966)

6. The production of quantitative economic statements that either explain the behaviour of variables we have already seen, or forecast (ie. predict) behaviour that we have not yet see, or both (Christ 1966)

7. Econometric is the art and science of using statistical methods for the measurement of economic relations (Chow, 1983).
4.3 Need for Econometrics

Economic theory makes statements or hypotheses that are mostly qualitative in nature. For example, Micro economic theory states that other things remaining the same, a reduction in the price of a commodity is expected to increase the quantity demanded of that commodity. Thus economic theory postulates a negative or inverse relation between price and quantity. But the theory does not provide any numerical measure of the relationship between the two. It is the job of the econometrician to provide such numerical estimates. Econometrics give empirical content to most economic theory.

4.4 Econometrics and Mathematical Economics

Mathematical economics states economic theory in terms of mathematical symbols. There is no essential difference between economic theory and mathematical economics. Economic theory uses verbal exposition where as Mathematical Economics uses mathematical symbols. In Mathematical Economics equations are formed without regard to the measurability or empirical verifications of the theory.

Econometrics differs from mathematical economics. Econometricians use mathematical equations but put these equations in such a way that they can be empirically tested. Econometric methods are designed to take into account random disturbances which create deviations from the exact behavioral pattern suggested by economic theory and mathematical economics.

4.5 Econometrics and Statistics

Economic statistics is mainly concerned with collecting, processing and presenting economic data in the form of charts and tables. It is mainly a descriptive aspect of economics. It does not provide explanation of the development of the various variables and it does not provide measurement of the parameters of economic relationship.

Mathematical statistics deals with methods of measurement which are developed on the basis of controlled experiments in laboratories. Statistical methods of measurement are not appropriate for economic relationships which cannot be measured on the basis of evidence provided by controlled experiments.

Econometrics uses statistical methods after adapting them to the problems of economic life. There adapted statistical methods are called econometric methods. The Econometricians like the meteorologists generally depends on data that cannot be controlled directly.

4.6 Goals of Econometrics

There are three main goals:

1. Analysis- the testing of economic theory
2. Policy making -supplying numerical estimates which can be used for decision making
3. Forecasting – using numerical estimates to forecast future values.
1. Analysis: Testing Economic Theory

The earlier economic theories started from a set of observations concerning the behavior of individuals as consumers or producers. Some basic assumptions were set regarding the motivations of individual economic units. From these assumptions the economists by pure logical reasoning derive some general conclusion regarding the working process of the economic system. Economic theories thus developed in an abstract level were not tested against economic reality. No attempt was made to examine whether the theories explained adequately the actual economic behavior of individuals.

Econometrics aims primarily at the verifications of economic theories. That is obtaining empirical evidence to test the explanatory power of economic theories. To decide how well they explain the observed behavior of the economic units.

2. Policy Making

Various econometric techniques can be obtained in order to obtain reliable estimates of the individual co-efficient of economic relationships. The knowledge of numerical value of these coefficients is very important for the decision of the firm as well as the formulation of the economic policy of the government. It helps to compare the effects of alternative policy decisions.

For eg. If the price elasticity of demand for a product is less than one (inelastic demand) it will not benefit the manufacturer to decrease its price, because his revenue would be reduced. Since econometrics can provide numerical estimate of the co-efficient of economic relationships it becomes an essential tool for the formulation of sound economic policies.

3. Forecasting Future Values

In formulating policy decisions it is essential to be able to forecast the value of the economic variables. Such forecasts will enable the policy makers to make efficient decision. In formulating policy decisions, it is essential to be able to forecast the value of the economic magnitudes. For example, what will be the demand for food grains in India by 2020? Estimates about this are essential for formulating agriculture production policies. Similarly, what will be the impact of a rise in deposit rate in share market and so on? It is known that if the bank deposit rates go up, day to day demand for shares will come down. Econometric tools help in such decision makings.

4.7 Scope of Econometrics

To make the meaning of econometrics more clear and detailed, it is appropriate to quote Frisch (1933) in full. “……econometrics is by no means the same as economic statistics. Nor is it identical with what we call general economic theory, although a considerable portion of this theory has a definitely quantitative character. Nor should econometrics be taken as synonymous with the application of mathematics to economics. Experience has shown that each of these three view points, that of statistics, economic theory, and mathematics, is necessary, but not by itself a sufficient, condition for a real understanding of the quantitative relations in modern economic life.
It is this unification of all three that is powerful. And it is this unification that constitutes econometrics”.

Econometric methods are statistical methods specifically adapted to the peculiarities of economic phenomena. The most important characteristic of economic relationship is that they contain a random element which is ignored by economic theory and mathematical economics. Econometrics has developed a method for dealing with the random component of economic relationship.

For example, Economic theory postulates that the demand for a commodity depends on its price, price of other commodities, income of the consumer, and tastes. This is an exact relationship, because it implies that demand is completely determined by the above four factors. The demand equations can be written as:

\[ Q = b_0 + b_1p + b_2p_0 + b_3y + b_4t \]

Where \( Q \) = quantity demanded of a particular commodity
\( P \) = price of the commodity
\( P_0 \) = price of other commodities
\( Y \) = consumers income
\( T \) = tastes
\( b_0, b_1, b_2, b_3, b_4 \) = coefficients of the demand equation.

The above equation is exact, because it implies that the only determinants of the quantity demanded are the four factors in the R.H.S. But other factors can affect demand which is not taken into consideration. For example, Invention of a new product, a war, changes in law, change in income distribution etc. In econometrics the influence of these other factors is taken into account by the introduction of a random variable. The demand functions can then be written as:

\[ Q = b_0 + b_1p + b_2p_0 + b_3y + b_4t + u \]

Where \( u \) stands for the random factors which affect demand.

Econometrics presupposes the existence of a body of economic theory. Economic theory comes first which is then tested with the application of econometric techniques. In testing a theory, mathematical formulation of the theory is first made \( Q = b_0 + b_1p + b_2p_0 + b_3y + b_4t + u \). The next step is to compare observational data with the mathematical model. This is to establish whether the theory can explain the actual behavior of the economic units. If the theory is compatible with actual data, the theory is accepted as valid. If the theory is incompatible with the observed data, the theory can be rejected or modified.
4.8 Methodology of Econometrics

Traditional econometric methodology proceeds along the following steps

1. **Statement of theory or hypothesis.**
2. **Specification of the mathematical model of the theory**
3. **Specification of the statistical, or econometric, model**
4. **Obtaining the data**
5. **Estimation of the parameters of the econometric model**
6. **Hypothesis testing**
7. **Forecasting or prediction**
8. **Using the model for control or policy purposes.**

**1. Statement of the theory or hypothesis**

Keynes postulated that the marginal property to consume, the rate of change of consumption for a unit change in income is greater than zero but less than one.

**2. Specification of the mathematical model of consumption.**

Keynesian consumption function can be mathematically expressed as: \( Y = \beta_1 + \beta_2 X \),

where \( Y \) = consumption expenditure, \( X \) = income and \( \beta_1 \) and \( \beta_2 \) the parameters of the model are respectively the intercept and slope coefficients. The slope coefficient \( \beta_2 \) measures the MPC. This equation which states that consumption is linearly related to income is an e.g. of the mathematical model of the relationship between consumption and income that is called the consumption function. The variable on the L.H.S in the dependent variable and the variables on the R.H.S. are the independent or explanatory variables. In the Keynesian consumption function, consumption expenditure is the dependent variable and income is the explanatory variable.

**3. Specification of the econometric model of consumption**

Mathematical model assumes that there is an exact or deterministic relationship between consumption and income. But relationship between economic variables is generally inexact. For e.g. If a sample of 500 families in taken and data plotted on a graph with consumption expenditure on the vertical axis and disposable income on the horizontal axis, we cannot expect all 500 observations to lie exactly on the straight line of eqn (1). This is because in addition to income, other variables affect consumption expenditure. For e.g., size of the family, ages of the members of the family etc can affect consumption.

The econometric model can be written as:
\[ Y = \beta_1 + \beta_2 X + u \]

Where \( u \) is the disturbance or error term, or a random (stochastic) variable. The econometric consumption function hypothesize that the dependent variable (consumption) is linearly related to the explanatory variable (income), but that the relationship between the two is not exact; it is subject to individual variations.

4. Obtaining Data

Estimations are possible only if data are gathered. Data can be collected either by census method or sample method. Important sampling methods used are simple random sample, stratified sample, systematic sample, multistage sampling, cluster sampling and quota sampling. Similarly, data are classified into primary data, secondary data, time series data, cross section data and pooled data. To estimate the numerical values of \( \beta_1 \) and \( \beta_2 \), data is needed. Three types of data are available for empirical analysis, time series, cross sectional and pooled data. In econometric models, the distinction between time series data and cross section data are important. To make its distinction clear, let us consider the following example,

<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>Sales</td>
<td>15</td>
<td>14</td>
<td>17</td>
<td>14</td>
<td>12</td>
<td>14</td>
<td>17</td>
<td>14</td>
<td>12</td>
</tr>
</tbody>
</table>

A casual look into the data set gives an impression that it belongs to time series, because it is ordered in time. But the given set is neither time series nor cross section.

**Time Series Data** give information about the numerical values of variables from period to period. The data can be collected at regular time intervals (daily, weekly, monthly, annual etc). For a data set to be time series, there are two conditions. Data collection interval should be equal and gather information on a single entity. The given set of data does not obey these conditions and hence not time series. But if we are provided with sales data for a few years, with regular intervals, on year, six months etc, definitely they constitute time series data.

**Cross Section Data** gives information on variables concerning individual agents (consumers or producers) at a given point of time. For e.g. a cross section sample of consumers is a sample of family budgets showing expenditures in various commodities by each family, as well as information on family income, family composition and other demographic, social or financial characteristics. When we gather information on multiple entities at a point of time, it is called cross section data. For example, if we are gathering details of income, savings, education, occupation etc of a group of 35 persons at a point of time, it is the best example of cross section data. In other words, survey data are broadly cross section data. In short, time series data is gathered at an interval of time while cross section data are gathered at a point of time. The classification of time series
and cross section data are important because, the use of appropriate techniques depends on the
nature of the data, whether it is time series or cross section.

Another set of data used in econometric modeling is **pooled data**. Pooled data, in a simple way
is the integration or mixing of time series and cross section data. But the treatment pooled data set
is little complicated. On the pooled data, all elements of both the series and cross sectional data
used. Data in real terms (i.e. they are measured in constant prices) is used. These data are plotted
in a graph where y variable is the aggregate consumption expenditure and X variable is GDP, a
measure of aggregate income.

5. **Estimation of the econometric model**

The numerical estimates of parameters can be found. Using this, the consumption functions can
be shown empirically. For estimating the parameters the technique of regression analysis is used.
If the estimates of $\beta_1$ and $\beta_2$ are respectively -231.8 & 0.7194, then the estimated consumption
functions is $\hat{Y} = -231.8 +0.7194x$ .The hat on y indicates that it is an estimate. The equation shows
that MPC = 0.72. This suggests that an increase in real income of one dollar, led on average, to
increases of about 72 cents in real consumption expenditure.

6. **Hypothesis Testing**

Keynesian theory says that MPC is positive but less than one. In the e.g. used, MPC was found
to be 0.72. If 0.72 is statistically less than one, then Keynesian theory can be supported. Such
confirmation or rejection of economic theories on the basis of sample evidence is known as
statistical inference (hypothesis testing).

7. **Forecasting or prediction**

Forecasting is one of the prime aims of econometric analysis and research. The forecasting
power will be based on the stability of the estimates, their sensitivity to changes in the size of the
sample. We must establish whether the estimated function performs adequately outside the sample
of data whose average variation it represents. One way of establishing the forecasting power of a
model is to use the estimates of the model for a period not included in the sample. The estimated
value or forecast value is compared with the actual or realized magnitude of the relevant dependent
variable. Usually there will be a difference between the actual and the forecast value of the
variable, which is tested with the aim of establishing whether it is statistically significant. If, after
carrying out the relevant test of significance, we find that the difference between the realized value
of the dependent variable and that estimated from the model is statistically significant, we conclude
that the forecasting power of the model is poor. Another way of establishing the stability of the
estimates and the performance of the model outside the sample of data, from which it has been
estimated, is to re estimate the function with an expanded sample that is a sample including
additional observations. The original estimates will normally differ from the new estimates. The
difference is tested for statistical significance with appropriate methods.
If the model confirms the hypothesis or theory under consideration, it can be used to predict the future values of the dependent variable \( y \) on the basis of known or expected future values of the explanatory variable \( x \). For e.g. suppose the real GDP is expected to be 6000 billion in 2010. Then the forecast of consumption expenditure can be estimated as

\[
Y = -238 + 0.7196 \times (6000)
\]

\[
= 4084.6
\]

The income multiplier is defined as:

\[
M = \frac{1}{1 - mpc} = \frac{1}{1 - 0.72} = 3.57
\]

This shows that an increase of a dollar investment will eventually lead to about four times increase in income. Thus, a quantitative estimate of MPC provides valuable information for policy making.

8. Policy implications

Using the Keynesian consumption function as e.g. Suppose the government believes that an expenditure level of 4000 billion dollars will reduce the level of unemployment. We can estimate the level of income which produces the targeted amount of consumption expenditure.

\[
4000 = -231.8 + 0.7194 \times x
\]

\[
x = \frac{4000 + 231.8}{0.7194}
\]

\[
x = 5882 \text{ approximately}
\]

That is, an income level of 5882, given a MPC of about 0.72 will produce expenditure equal to 4000 billion dollars.

4.9 Desirable Properties of an Econometric Model

An econometric model is a model whose parameters have been estimated with some appropriate econometric technique. The goodness of an econometric model is judged according to the following desirable properties.

1. Theoretical plausibility – the model should be compatible with the postulates of the economic theory.

2. Explanatory ability – The model should be able to explain the observations of the actual world. It must be consistent with the observed behaviour of the economic variables.

3. Accuracy of the estimates of the parameters – The estimates of the coefficient should be accurate in the sense that, they should approximate as best as possible, the true parameters.
of the structural model. The estimates should possess properties like unbiasedness, consistency and efficiency

4. Forecasting ability – The model should produce satisfactory predictions of future values of the dependent variable.

5. Simplicity – the model should represent the economic relations with maximum simplicity. The fewer the equations and simpler their mathematical form, the better the model is considered.

4.10 Types of Econometrics

Econometrics may be divided into two broad categories. Theoretical econometrics and applied econometrics.

Theoretical econometrics is concerned with the development of appropriate method for measuring economic relationships specified by econometric models. For e.g. one of the methods used extensively is the principle of least squares.

In applied econometrics, the tools of theoretical econometrics, is used to study some special area of economics and business such as the production function, investment function, demand & supply functions etc.

4.11 Uses of Econometrics

1. Econometrics is widely used in policy formulation

   For eg. Suppose the government wants to devalue its currency to correct the balance of payment problem. For estimating the consequences of devaluation, the price elasticity of imports and exports is needed. If imports and exports are inelastic then devaluation will not produce the necessary change. If imports and exports are elastic then the BOP of the country will improve by devaluation. Price elasticity can be estimated with the help of demand function of import and export. An econometric model can be built through which the variables can be estimated.

   2. Econometrics helps the producers in making rational calculations.

   3. Econometrics is also useful in verifying theories.

   4. Studies of econometrics mainly consist of testing of hypothesis, estimation of the parameters and ascertaining the proper functional form of the economic relations.

4.12 Limitations of Econometrics

Econometrics has come a long way over a relatively short period of time. Important advances have been made in the compilation of data, development of concepts, theories and tools for the construction and evaluation of a wide variety of econometric models. Applications of econometrics can be found in almost every field of economics. Nowadays, even there is a tendency to use econometric tools in certain other sciences like sociology, political science, agriculture and
management. Econometric models have been used frequently by government departments, international organizations and commercial enterprises. At the same time, experience has brought out a number of difficulties also in the use of econometric tools. The important limitations are,

1. Quality of data: Econometric analysis and research depends on intensive data base. One of the serious problems of Indian econometric research is non availability of accurate, timely and reliable data.

2. Imperfections in economic theory: Earlier it was felt that the economic theory is sufficient to provide base for model building. But later it was realized that many of the economic theories are illusory because they are based on the assumption of ceteris paribus and hence models can not fully accommodate the dynamic forces behind a phenomena.

3. There are institutional features and accounting conventions that have to be allowed for in econometric models but which are either ignored or are only partially dealt with at the theoretical level.

4. Any economic phenomenon is influenced by social, cultural, political, physiological and even physical factors. These factors can not be easily quantified. Even if quantified, they may not be capable of explaining the phenomenon properly. For example, it is said that the intelligentsia of Indian planners gave birth to very beautiful mathematical models, but they forgot to feed the hungry masses.

5. The method of econometrics can be applied only to quantifiable phenomenon. It is difficult to estimate the values of parameters in the case of qualitative problems.

6. The main difficulty with econometrics is that statistical tools are used to estimate parameters. Statistical methods are based upon certain assumptions which may not be consistent with economic data.

7. Another limitation is that econometric models are abstract in nature. They do not help in forming moral judgments. But in policy formulation often moral judgments are necessary.

8. Econometric methods are time consuming, tedious and complex in nature.

**QUESTIONS**

**A. Multiple Choices**

1. The term ‘econometrics’ was coined by
   (a) Marsahll  (b) Pawel  (C) Ragnar Frisch  (d) Clompa

2. Error term serves the purpose of………………… assumption in economics
   (a) Dynamic  (b) static  (c) comparative  (d) none of the above

3. Econometrics model is …………..model
(a) exogenous (b) endogenous (c) identified (d) either exogenous or endogenous

4. The starting point of econometric analysis is
   (a) model specification    (b) formulation of alternative hypothesis
   (c) formulation of null hypothesis (d) collection of data

5. Regressor refers to
   (a) independent variable (b) dependent variable  (c) error term (d) dummy variable

6. In perfect linear model, we assume that regression coefficient remains………..
   (a) variable until some point (b) variable through out (c) constant to some point
   (d) constant through out

7. In econometric models, t+1 indicates,
   (a) net addition (b) current value with some fluctuations  (c) expected value (d) none of these

8. Quota sample is……………….sample
   (a) probability sample (b) non probability sample (c) convenientsample (d) judgment sample

9. When a north Indian town data and south Indian data are totaled, it leads to the problem of
   …………………aggregation.
   (a) national    (b) regional    (c) spatial    (d) heterogeneous

10. By theoretical plausibility, we mean,
    (a) ability to explain economic theory    (b) ability to prove economic theory
    (c) ability to validate economic theory    (d) all of the above

**Answers**  (1) C  (2) D  (3) D  (4) C  (5) A  (6) D  (7) C  (8) B  (9) C  (10) A.

**B. Very short answers**

1. Distinguish between mathematical economics and econometrics

2. Define econometrics according to Christ

3. What are the goals of econometrics
4. Distinguish between a mathematical model and econometric model
5. Distinguish between time series and cross section data
6. Give any two desirable properties of an econometric model

C. Short answers
1. Justify the need of a stochastic error term
2. Explain the sources of hypothesis formulation
3. Explain a priori criterion for evaluating an econometric model

D. Essay questions
1. Explain the econometric methodology in detail with examples
2. Examine the scope of econometrics.
MODULE V

THE LINEAR REGRESSION MODEL


5.1 Regression Analysis

The term regression was introduced by Francis Galton. Regression analysis is concerned with the study of the dependence of one variable (dependent variable) on one or more other variables (explanatory variables) with a view to estimating the average (mean) value of the former in terms of known (fixed) values of the latter. Galton found that, although there was a tendency for tall parents to have tall children and for short parents to have short children, the average height of children born of parents of a given height tended to more or regress towards the average height in the population as a whole. In other words, the height of the children of unusually tall or unusually shorts parents tends to more towards the average height of the population. In the modern view of regression, the concern is with finding out how the average height of sons changes, given the fathers height. Regression analysis is largely concerned with estimating and/or predicting the (population) mean value of the dependent variable on the basis of the known or fixed values of the explanatory variable.

5.2 Origin of the Linear Regression Model

There are different methods for estimating the coefficients of the parameters. Of these different methods, the most popular and widely used is the regression technique using Ordinary Least Square (OLS) method. This method is used because of the inherent properties of the estimates derived using this method. But, first let us try to understand the rationale of this method. For this purpose, let us go back to the demand theory as well as the consumption function which we discussed in the earlier chapter. Demand theory says that there is a negative relation between price and quantity demanded \textit{ceteris paribus}. In the case of consumption function, there is a positive relation between consumption expenditure and income. There are three important questions here.

1. Which is the dependent variable and which is the independent variable?

2. Which is the appropriate mathematical form which explains the phenomenon?

3. What is the expected sign and magnitude of the coefficients?

In order to answer these questions, the theory will give the necessary support. In the case of demand equation, quantity demanded is the dependent variable, and price is the independent variable. Economic theory does not discuss the choice between single equation models or simultaneous equation models to discuss the relationship. So naturally we may assume that the relation is explained with the help of single equation, that too assuming a linear relation. As far as
the sign and magnitude of the coefficients are concerned, in the equation, \( D = \alpha + \beta P + U \), \( \alpha \) can take any value but preferably zero or positive. It actually shows the quantity demanded at price zero. So chances of demanding negative quantity is very rare and hence if we get negative quantity, it can be approximated to zero. In the case of \( \beta \), it can be positive or negative. But normally it will be negative assuming that the commodity demanded is a normal good. Of course, elasticity nature of the commodity also influences the magnitude and nature of this value.

In the case of consumption function, consumption is the dependent variable and income is the independent variable. Whether the relation is linear or non linear, is a debatable issue. For instance, psychological law of Keynes suggests that when income increases, consumption also increases, but less than proportionate. So assuming that consumption and income are linearly related is in one way, over simplification. But for the time being let us assume so just for explanatory purpose. Regarding the sign and magnitude of parameters \( \alpha \) and \( \beta \). There is some meaning and interpretation. \( \alpha \) represents the consumption when income takes the value zero, that is, according to theory, it is autonomous consumption. Similarly, \( \beta \) is nothing but the value of marginal propensity to consume which is normally less than 1 and cannot be negative.

Based on the above discussed rationale and logic, let us rewrite the demand equation as \( D = \alpha + \beta P + U \), where \( D \) is the quantity demanded, \( P \) is price, \( \alpha \) and \( \beta \) are the parameters to be estimated. In order to estimate these parameters, we use Ordinary Least Square (OLS) method. Once we plot this on a graph, we will be able to get the deviations between actual and estimated observations, popularly called as errors. Naturally, a rational decision is to minimize these errors. Thus from all possible lines, we choose the one for which the deviations of the points is the smallest possible. The least squares criterion requires that the regression line be drawn in such a way, so as to minimize the sum of the squares of the deviations of the observations from it. The first step is to draw the line so that the sum of the simple deviations of the observations is zero. Some observations will lie above the line and will have a positive deviation, some will lie below the line, in which case, they will have a negative deviation, and finally the points lying on the line will have a zero deviation. In summing these deviations the positive values will offset the negative values, so that the final algebraic sum of these residuals will equal zero. Mathematically, \( \sum e = 0 \). Since the sum total of deviations is 0, it can not be minimized as such. So we try to square the deviations and minimize the sum of the squares. \( \sum e^2 \). Thus we call this method as least square method,

### 5.3 Population Regression Function (PRF)

Mathematically a population regression function (PRF) or Conditional Expectation Function (CEF) can be defined as the average value of the dependent value for a given value of the explanatory or independent variable. In other words, PRF tries to find out how the average value of the dependent variable varies with the given value of the explanatory variable. On the other hand, when we estimate the average value of the dependent variable with the help of a sample, it is called stochastic sample regression function (SRF).

\[
E(Y \mid X_i) = f(X_i)
\]
Where; \( f(X_i) \) denotes some function of the explanatory variable \( X \).

\( \text{E}(Y \mid X_i) \) is a linear function of \( X_i \). This is known as the conditional expectation function (CEF) or population regression function (PRF). It states merely that the expected value of the distribution of \( Y \) given \( X_i \) is functionally related to \( X_i \). In simple terms, it tells how the mean or average response of \( Y \) varies with \( X \). For example, an economist might posit that consumption expenditure is linearly related to income. Therefore, as a first approximation or a working hypothesis, we may assume that the PRF \( \text{E}(Y \mid X_i) \) is a linear function of \( X_i \),

\[
\text{E}(Y \mid X_i) = \beta_1 + \beta_2 X_i
\]

Where; \( \beta_1 \) and \( \beta_2 \) are unknown but fixed parameters known as the regression coefficients; \( \beta_1 \) and \( \beta_2 \) are also known as intercept and slope coefficients, respectively.

We can express the deviation of an individual \( Y_i \) around its expected value as follows: \( u_i = Y_i - \text{E}(Y \mid X_i) \) or

\[ Y_i = \text{E}(Y \mid X_i) + u_i \]

where the deviation \( u_i \) is an unobservable random variable taking positive or negative values. Technically, \( u_i \) is known as the stochastic disturbance or stochastic error term.

We can say that the expenditure of an individual family, given its income level, can be expressed as the sum of two components: (1) \( \text{E}(Y \mid X_i) \), which is simply the mean consumption expenditure of all the families with the same level of income. This component is known as the systematic, or deterministic, component, and (2) \( u_i \), which is the random, or nonsystematic, component is a surrogate or proxy for all the omitted or neglected variables that may affect \( Y \) but are not (or cannot be) included in the regression model.

If \( \text{E}(Y \mid X_i) \) is assumed to be linear in \( X_i \), it may be written as

\[
Y_i = \text{E}(Y \mid X_i) + u_i = \beta_1 + \beta_2 X_i + u_i
\]

### 5.4 Sample Regression Function (SRF)

Since the entire population is not available to estimate \( y \) from given \( x_i \), we have to estimate the PRF on the basis of sample information. From a given sample we can estimate the mean value of \( y \) corresponding to chosen \( x_i \) values. The estimated PRF value may not be accurate because of sampling fluctuations. Because of this only an approximate value of PRF can be obtained. In general, we would get \( N \) different sample regression function (SRFs) for \( N \) different samples and these SRFs are not likely to be the same.

We can develop the concept of the sample regression function (SRF) to represent the sample regression line.

\[
\hat{Y} = \beta_1 + \beta_2 X_i
\]
Where \(^{\hat{Y}}\) is read as “Y-hat” or “Y-cap”

\[^{\hat{Y}}_{i} = \text{estimator of } E(Y \mid X_i)\]

\[^{\hat{\beta}_1} = \text{estimator of } \beta_1\]

\[^{\hat{\beta}_2} = \text{estimator of } \beta_2\]

Note that an estimator, also known as a (sample) statistic, is simply a method that tells how to estimate the population parameter from the information provided by the sample at hand.

We can express the SRF in its stochastic form as follows:

\[ Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \]

where, in addition to the symbols already defined, \(\hat{u}_i\) denotes the estimate of the error term.

### 5.5 Significance of the Stochastic Error Term

The disturbance term \(u_i\) is a surrogate for all those variables that are omitted from the model but that collectively affect \(Y\).

1. Vagueness of theory

The theory determining the behavior of \(Y\) may be incomplete. We might know for certain that weekly income \(X\) influences weekly consumption expenditure \(Y\), but we might be ignorant or unsure about the other variables affecting \(Y\). Therefore \(u_i\) maybe used as a substitute for all the excluded or omitted variables from the model.

2. Unavailability of data

Even if we know what some of the excluded variables are we may not have quantitative information about these variables. For example, in principle we could introduce family wealth as an explanatory variable in addition to the income to explain family consumption expenditure. But unfortunately, information on family wealth generally is not available.

3. Core variables versus peripheral variables

Assume in our consumption income example that besides income \(X_1\), the number of children per family \(X_2\), sex \(X_3\), religion \(X_4\), education \(X_5\), and geographical region \(X_6\) also affect consumption expenditure. But it is quite possible that the joint influence of all these variables may be so small that it need not be introduced in the model. Their combined effect can be treated as a random variable \(u_i\).

4. Intrinsic randomness in human behavior

Even if all the relevant variables affecting \(y\) are introduced into the model, there may be variations due to intrinsic randomness in individual which cannot be explained. The disturbance term \(u_i\) also include this intrinsic randomness.
5. Poor proxy variables

Although the classical regression model assumes that variables y and x are measured accurately, it is possible that there may be errors of measurement. Variables which are used as proxy may not provide accurate measurement. The disturbance term u can also be used to include errors of measurement.

6. Principle of parsimony

Regression model should be formulated as simple as possible. If the behavior of y can be explained with the help of two or three explanatory variables then more variation need not be included in the model. Let u, represent all other variables. This does not mean that relevant and important variables should be excluded to keep the regression model simple.

7. Wrong functional form

Even if we have theoretically correct variables exploring a phenomenon and even if it is possible to get data on these variables, very often the functional relationship between the dependent and independent variable may be uncertain. In two variable models functional relation can be ascertained with the help of scattergram. But in multiple regression model it is not easy to determine the, approximate functional form. Scattergram cannot be visualised in multi dimensional form. For all these reasons, the stochastic disturbance u, assumes an extremely critical role in regression analysis.

5.6 Assumptions of Classical Linear Regression Model

1. U is a random real variable. The value which may assume in any one period depends on chance. It may be positive, zero or negative. Each value has a certain probability of being assumed by U in any particular instance.

2. The mean value of U in any particular period is zero. If we consider all the possible values of U, for any given value of X, they would have an average value equal to zero. With this assumption we may say that \( Y = \beta X + U \) gives the relationship between X and Y on the average. That is, when X assumes the value X1, the dependent variable will on the average assume the value Y1, although the actual value of Y observed in any particular occasion may display some variation.

3. The variance of U is constant in each period. The variance of U about its mean is constant at all values of X. In other words, for all values of X, the U will show the same dispersion round their mean.

4. The variable U has a normal distribution

5. The random terms of different observations are independent. This means that all the covariance of any U (ui) with any other U (uj) are equal to zero

6. U is independent of the explanatory variables
The above mentioned assumptions are really classic to regression estimations and make the method OLS efficient.

There are a few other assumptions also used in OLS estimated. They are,

i  The explanatory variables are measured without error. In other words, the explanatory variables are measured without error. In the case of dependent variable, error may or may not arise.

ii  The explanatory variables are not perfectly linearly correlated. If there is more than one explanatory variable in the relationship, it is assumed that they are not perfectly correlated with each other. More specifically, we are assuming the absence of multicollinearity.

iii There is no aggregation problem. In the previous chapter, we discussed aggregation over individuals, time, space and commodities. So we assume the absence of all these problems.

iv The relationship being estimated is identified. This means that we have to estimate a unique mathematical form. There is no confusion about the coefficients and the equations to which it belong.

v The relationship is correctly specified. It is assumed that we have not committed any specification error in determining the explanatory variables, in deciding the mathematical form etc.

5.7 The Method of Ordinary Least Squares

The method of ordinary least squares is attributed to Carl Friedrich Gauss, a German mathematician. The method of least squares has some very attractive statistical properties that have made it one of the most powerful and popular methods of regression analysis. To understand this method, we first explain the least squares principle.

Given the PRF:

\[ Y_i = \beta_1 + \beta_2 X_i + u_i \]

But it is not easy to estimate PRF, we have to estimate it from the SRF:

\[ Y_i = \hat{\beta}_1 + \hat{\beta}_2 X_i + \hat{u}_i \]

where \( \hat{Y}_i \) is the estimated value of \( Y_i \). From the equation of SRF we can write:

\[ \hat{u}_i = Y_i - \hat{Y}_i = Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i \]

Which shows that the \( \hat{u}_i \) (the residuals) are simply the differences between the actual and estimated \( Y \) values. Given \( n \) pairs of observation on \( Y \) and \( X \) SRF can be determined so that its value is as close as possible to the actual \( Y \). For this the least square estimate is used such that SRF is equal to \( \sum \hat{u}_i^2 = \sum (Y_i - \hat{Y}_i)^2 = \sum (Y_i - \hat{\beta}_1 - \hat{\beta}_2 X_i)^2 \)
The principle or the method of least squares chooses $\hat{\beta}_1$ and $\hat{\beta}_2$ in such a manner that, for a given sample the error term is made as small as possible. In other words, for a given sample, the method of least squares provides us with unique estimates of $\beta_1$ and $\beta_2$ that give the smallest possible value of the error term. The sum of squared residual deviations is to be minimised with respect to parameters. So we use little amount of differential calculus and applying the minimization rules, the first derivative should be equal to zero and second derivative should be greater than zero, we finally arrive at two equations, popularly known as normal equations. The equations are,

$$N \hat{\beta}_1 + \hat{\beta}_2 X = \Sigma y$$

$$\hat{\beta}_1 \Sigma X + \hat{\beta}_2 \Sigma X^2 = \Sigma XY, \text{ N refers to number of observations}$$

Using these two equations, we can easily estimate the parameters. The estimators so obtained are called least square estimators, for they are derived from the least square principle.

5.8 Properties of OLS estimate

The least square estimates are BLUE (best, linear and unbiased), provided that the random term $U$ satisfies some general assumptions, namely that the $U$ has zero mean and constant variance.

1. It is linear, that is, a linear function of a random variable.
2. It is unbiased, that is, its average or expected value is equal to the true value of the coefficient.
3. It has minimum variance in the class of all such linear unbiased estimators.

An unbiased estimator with the least variance is known as an efficient estimator. In one way, this is the gist of the famous Gauss Markov theorem which can be stated as “given the assumptions of the classical linear regression model, the least squares estimators, in the class of unbiased linear estimators, have minimum variance, that is, they are BLUE”.

5.9 Test of Significance of Regression coefficients

Before discussing the conventional tests used in econometric analysis, it is appropriate at juncture to have little statistical theory and logic behind testing.

5.10 Statistical inferences

Statistical inferences are the area that describes the procedures by which we use the observed sample data to draw conclusions about the characteristics of the population on from which the data were generated. Statistical inference can be classified under two categories. Classical inferences and Bayesian inferences. Classical inference is based on two promises. (i) The sample data constitute the only relevant information and (ii) the construction and assessment of the different procedures for inference are based on long run behavior under essentially similar circumstances. In Bayesian inference we combine sample information with other prior information.
Classical inference constitutes two steps (i) estimation and (ii) testing of hypotheses.

(i) Estimation

There are two types of estimations, point estimation and interval estimation. Let be a parameter (mean, variance or any other moment) in a population probability distribution from which we have drawn a sample of size “n” denoted by \(x_1, x_2, x_3, \ldots, x_n\). In point estimation, \(\theta\) is estimated as a function of these sample observations denoted by \(x_1, x_2, \ldots\) this function is called an estimator. In specific case, when the function is determined by a numerical value, it is called an estimator of . Thus, \(\bar{X} = \frac{1}{n}\sum X\), the sample mean is an estimator of \(\theta\), the population mean and \(\bar{X} = 5\), is an estimate of \(\theta\) from a particular sample. Instead of one function of \(x_1, x_2, x_3, \ldots\), in the point estimation, here two functions are constructed from the sample observations and say that \(\theta\) lies between two points with a certain probability. This interval estimates are relevant in the testing of hypotheses. The key concept underlying the interval estimation is the notion of the sampling distribution of an estimator. For example, if a sample is drawn from a normal population with mean \(\mu\) and the sample mean \(\bar{X}\) is a point estimator of \(\mu\) then, \((\bar{X})N (\mu, \sigma^2/n)\), ie, the estimator from a sample is having a probability distribution (in this case, normal with mean \(\mu\), and variance \(\sigma^2/n\)). This enables us to construct the interval \((\bar{X}\pm 2\sigma/\sqrt{n})\) and claim that the probability distribution of having the true value of \(\mu\) within this interval is 0.95, using normal probability curve. The probability that the interval \((\bar{X}\pm 2\sigma/\sqrt{n})\), contains the true value of \(\mu\) is 0.95. This interval is called the confidential interval of size (confidence coefficient) of 0.95 or 95 per cent. This means that if we estimate \(\theta\) from repeated samples, we shall be getting all these values within this range in 95 cases out of 100. The complement of confidence coefficient of 0.95 is 0.05 or 5 per cent. It is denoted generally by \(\alpha\) and is called the level of significance in testing of hypotheses.

(ii) Testing of hypotheses

Let \(f(x)\) be a population density function of \(x\) with \(\theta\) as the parameter of the distribution. Let estimated be the point estimator obtained from a random sample of size “n” from this population. Our intention is to judge the value of \(\theta\), the population parameter on the basis of estimated . For example, can we say from the estimated that the value of \(\theta = \theta^*\), any specific value we guess, say 15. In other words, can we say that the sample we used have come from a population with \(\theta = \theta^*\). A statistical hypothesis is a statement about the value of some parameters in a population from which the sample is drawn and is denoted by \(H\). The hypothesis we intend to test is called a Null or maintained hypothesis and denoted by \(H_0\). Thus \(H_0: \theta = \theta^*\) is the null hypothesis. Complementary to this, we can state another hypothesis that \(\theta \neq \theta^*\), which is called the alternative hypothesis denoted by \(H_1\). In testing of hypothesis, we test \(H_0: \theta = \theta^*\) against the alternative \(H_1 \neq \theta^*\).

Two possibilities of making errors exist. A null hypothesis may be really true, but on the basis of test, we may conclude it is wrong and thus reject \(H_0\) when it is actually true. The error we commit in this process is called Type I error or \(\alpha\) error. Alternatively, a null hypothesis may be really wrong, but we may conclude on the basis of the test that it is true and thus we do not reject null hypothesis when it is actually wrong. This error is called Type II error or \(\beta\) error.
The test procedure, ideally, should be such that both Type I and Type II errors are eliminated or the probability of committing these errors is zero. We denote probability of committing Type I error by $\alpha$ and the probability of committing Type II error by $\beta$. Now $\alpha$ is called the level of significance and $(1-\beta)$ the power of the test.

5.11 Student t or Z test:

t test is applicable to a small sample and Z test is applicable to a large sample. These tests undergo a detailed testing procedure where we have to consider the degrees of freedom, level of significance, the choice between one tailed test/two tailed test and so on. All these testing procedures have already been explained above. In order to get the estimated values of $t$ or $z$, there is a short cut. In order to get the $t$ value corresponding to intercept, just divide the estimated intercept value by its respective standard error and also in order to get the $t$ value for the coefficient, just divide the estimated coefficient value with its standard error, ie, $\sqrt{\frac{\Sigma e^2}{(n-2)\Sigma x^2}}$. If the calculated $t$ value is greater than the table $t$ value, we reject the hypothesis that $X$ and $Y$ are independent. If on the other hand, if the calculated $t$ value is smaller than the table $t$ value, we accept the null hypothesis, ie, $X$ and $Y$ are in dependent.

5.12 Coefficient of Determination ($R^2$): A measure of goodness of Fit

The goodness of fit means how well the sample regression line fits the given data. If all the observation were to lie on the regression line, it indicates a perfect fit. But this happens very rarely. Generally, there will be some positive and some negative $\hat{u}_i$. The aim is to make the residuals around the regression live as small as possible. The coefficient of determination $R^2$ is a measure that shows how well the sample regression line fits the data.

After the estimation of the parameters and the determination of the least squares regression line, we need to know how good is the fit of this line to the sample observations of $Y$ and $X$, that is to say, we need to measure the dispersion of observations around the regression line. This knowledge is essential, because the closer the observations to the line, the better the goodness of fit, that is the better is the explanation of variations of $Y$ by the changes in the explanatory variables. In order to measure this, we use coefficient of determination method. Coefficient of determination shows the percentage of the total variation of the dependent variable that can be explained by the independent variable. In other words, coefficient of determination is said to be the explanatory power of the model and is defined as,

\[ R^2 = 1 - \frac{\Sigma e^2}{\Sigma y^2} \text{ where } y = \text{ mean of } Y \]

The value of $R^2$ ranges between 0 and 1. If the value is exactly equal to 1, it is a case of exact relation and error is zero. This is practically impossible in social science. In majority of cases, the value will vary from 0.6 to 0.8.

5.13 Properties

1. It is a non negative quantity
2. Its limits are $0 \leq R^2 \leq 1$. When $R^2=1$, it means a perfect fit. When $R^2=0$, there is no
relationship between the dependent variable and explanatory variable.

3. A quantity closely related to but conceptually very much different from $R^2$ is the coefficient of correlation $r$. It is a measure of the degree of association between two variables.

**QUESTIONS**

A Multiple Choice

1. In an econometric model, $Y = \infty + \beta X$, $\infty$ shows,
   
   (a) Intercept of the equation  
   (b) Slope of the equation  
   (c) Average value of $Y$ for average value of $X$  
   (d) Rate of change

2. Error term indicates
   
   (a) Fluctuations in the given data  
   (b) Variations  
   (c) Random variations  
   (d) Explained variation

3. Among the following, which is an assumption of OLS
   
   (a) The explanatory variables are measurable  
   (b) The relationship being estimated is identified  
   (c) error term and independent variables are related  
   (d) error term and independent variables are linearly related

4. Linearity means
   
   (a) The OLS estimates are linear function of random variable  
   (b) The OLS estimates are function of variable  
   (c) The OLS estimates are function of random variable  
   (d) The OLS estimates has minimum variance

5. The property of average or expected value is equal to true value of the coefficient is the property of,
   
   (a) zero variance  
   (b) minimum variance  
   (c) zero mean  
   (d) minimum mean

6. The power of a statistical test is defined as,
   
   (a) $1 - \beta$  
   (b) $1 + \beta$  
   (c) $1$  
   (d) $\beta$

7. Standard error is defined as,
   
   (a) standard deviation of the sampling distribution  
   (b) standard deviation of the population  
   (c) variance of the sampling distribution  
   (d) variance of the population

8. Coefficient of determination shows
   
   (a) the percentage of the total variation in the dependent variable that can be explained by the independent variable  
   (b) the percentage of the variation in the dependent variable that can be explained by the independent variable  
   (c) none of the above
9. Student t test is preferred in the case of a,
   (a) small sample  (b) large sample  (c) when sample is below 50  (d) when sample is above 50

10. Cobb Douglas production function is an example of:
    (a) linear model  (b) double log model  (c) lin log model  (d) log lin model


B. Very short answers

1. What is specification bias?
2. What is a scatter diagram? What are its uses?
3. State BLUE
4. Distinguish between population regression function and sample regression function
5. Distinguish between type I and type II errors
6. Distinguish between confidence coefficient and power of a test

C. Short Answers

1. State the stochastic assumptions of OLS
2. Briefly explain Gauss Markov theorem
3. What are first order tests in econometric model evaluation?

D. Essay Questions

1. Explain the OLS method in detail.
2. Explain the assumptions of OLS model.