SCHOOL OF DISTANCE EDUCATION

CCSS UG PROGRAMME

MATHEMATICS (OPEN COURSE)

(For students not having Mathematics as Core Course)

MM5D03: MATHEMATICS FOR SOCIAL SCIENCES

FIFTH SEMESTER

STUDY NOTES

Prepared by:

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UNIVERSITY OF CALICUT

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Module-I

Chapter 1

Equations and Graphs

1.1 Equations

An *equation* is a mathematical statement that two things are equal. It consists of two expressions, one on each side of an 'equals'sign. for example $11 = 5 + 6$.This equation states that 11 is equal to the sum of 5 and 6, which is obviously true. In an equation, the left side is always equal to the right side.The most common equations contain one or more variables. If we let x stand for an unknown number, and write the equation $x = 12 + 5$. We know the left side and right side are equal, so we can see that x must be 12+5 or 17. This is the only value that x can have that makes the equation a true statement. We say that $x = 17$ 'satisfies'the equation. This process of finding the value of the unknowns is called *solving the equation.* We often say that we "solve for x"- meaning solve the equation to find the value of the unknown number x. Some equations have more than one solution. If the solution set includes all real numbers, the equation is an *identity.* If the solution set is empty, the equation has *no solution.* Two equations with identical solution sets are referred to as *equivalent.* An equation in which all variables are raised to the first power is known as a *linear equation.*

Example 1.1. In the mathematical expressions and statements listed below.

(a) $4x^2 + 7$

(b) $3x - 8 = 2y + 12$

(c) $6x + 7 = 19$

(d) $x^2 - 3x + 5$

(e) $x^2 - 6x + 5 = 5$

(b), (c) and (e) are equations. (b) is an equation in two variables x and y, but (c) is an equation in one variable x. (b) and (c) are linear equations. Note that (e) is not linear, since power of x in (e) is two.

The same quantity Q can be added, subtracted, multiplied or divided on both sides of the equation without effecting the equality, $Q \neq 0$ for division.

Properties of Equality

For all real numbers a, b, and c, if $a = b$,

(a) Addition Property : $a + c = b + c$

(b) Subtraction Property : $a - c = b - c$

(c) Multiplication Property : $ac = bc$

(d) Division Property : $a/c = b/c \quad (c \neq 0)$

Problem 1.1. Solve the equation $\frac{x}{2} - 3 = \frac{x}{3} + 1$

Solution. Solving an equation means we have to find the values of x which satisfies the given equation.We solve above problem in three steps.

(1) Move all terms with unknown variable to the left, here by subtracting $\frac{x}{3}$ from both sides of the equation, we get

$$\begin{aligned} \frac{x}{2} - 3 - \frac{x}{3} &= \frac{x}{3} + 1 - \frac{x}{3} \qquad (\textit{Subtraction Property}) \\ \frac{x}{2} - \frac{x}{3} - 3 &= 1 \end{aligned}$$

(2) Move terms without the unknown variable to the right, here by adding 3 to both sides of the equation.

$$\frac{x}{2} - \frac{x}{3} - 3 + 3 = 1 + 3 \qquad (\textit{Addition Property})$$

$$\frac{x}{2} - \frac{x}{3} = 4$$

(3) Simplify both sides,

$$\begin{aligned}
\frac{x}{2} - \frac{x}{3} &= 4 \\
\frac{3 \cdot x - 2 \cdot x}{2 \cdot 3} &= 4 \\
\frac{3x - 2x}{6} &= 4
\end{aligned}$$

Multiplying by 6 we get,

$$\begin{aligned}
6 \cdot \frac{3x - 2x}{6} &= 6 \cdot 4 \qquad (\textit{Multiplication Property}) \\
x &= 24
\end{aligned}$$

■

Note 1.1. Any equation which can be reduced to $0 = 0$ by steps using the properties of equality is an identity.

Note 1.2. Any equation that can be reduced to $a = 0$ through the properties of equality, when $a \neq 0$ has no solution.

Problem 1.2. Use the properties of equality to solve the following linear equations by moving all terms with the unknown variable to the left, all other terms to the right, and then simplifying :

(a) $4x + 9 = 7x - 6$

(b) $28 - 2x = 8x - 12$

(c) $9(3x + 4) - 2x = 11 + 5(4x - 1)$

Solution.

(a)

$$\begin{aligned}
4x + 9 &= 7x - 6 \\
4x - 7x &= -6 - 9 \\
-3x &= -15 \\
x &= 5
\end{aligned}$$

(b)

$$\begin{aligned} 28 - 2x &= 8x - 12 \\ -2x - 8x &= -12 - 28 \\ -10x &= -40 \\ x &= 4 \end{aligned}$$

(c)

$$\begin{aligned} 9(3x + 4) - 2x &= 11 + 5(4x - 1) \\ 27x + 36 - 2x &= 11 + 20x - 5 \\ 25x + 36 &= 6 + 20x \\ 25x - 20x &= 6 - 36 \\ 5x &= -30 \\ x &= -6 \end{aligned}$$

■

Problem 1.3. Solve the following equations:

(a) $5x - 39 = 5(x - 8) + 1$

(b) $8x - 13 = 8x + 9$

Solution.

(a)

$$\begin{aligned} 5x - 39 &= 5(x - 8) + 1 \\ 5x - 39 &= 5x - 40 + 1 \\ 5x - 39 &= 5x - 39 \\ 5x - 5x &= 39 - 39 \\ 0 &= 0 \end{aligned}$$

This means that any real number can be substituted for x and the equation will be valid. Therefore above equation is an identity.

(b)

$$\begin{aligned} 8x - 13 &= 8x + 9 \\ 8x - 8x &= 9 + 13 \\ 0 &= 22 \end{aligned}$$

Which is not true. Therefore there is no real number that, when substituted for x will make the equation valid. So the above equation has no solution.

■

Problem 1.4. In the following equations solve for x by clearing the denominator, that is, by multiplying both sides of the equation by the least common denominator as soon as is feasible.

(a) $\frac{x}{3} + \frac{x}{2} = 15$

(b) $\frac{8}{x-2} + \frac{15}{x-3} = \frac{21}{x-3} \quad (x \neq 2, 3)$

Solution.

(a)

$$\frac{x}{3} + \frac{x}{2} = 15$$

Multiplying both sides of the equation by 6,

$$\begin{aligned} 6 \cdot \left(\frac{x}{3} + \frac{x}{2}\right) &= 6 \cdot 15 \\ 2x + 3x &= 90 \\ 5x &= 90 \\ x &= \frac{90}{5} = 18 \end{aligned}$$

(b)

$$\frac{8}{x-2} + \frac{15}{x-3} = \frac{21}{x-3} \quad (x \neq 2, 3)$$

Multiplying both sides of the equation by $(x-2)(x-3)$,

$$\begin{aligned}
(x-2)(x-3)\left(\frac{8}{x-2}+\frac{15}{x-3}\right) &= \frac{21}{x-3}(x-2)(x-3)\\
8(x-3)+15(x-2) &= 21(x-2)\\
8x-24+15x-30 &= 21x-42\\
23x-54 &= 21x-42\\
23x-21x &= -42+54\\
2x &= 12\\
x &= 6
\end{aligned}$$

■

1.2 Cartesian Coordinate System

A *Cartesian coordinate system* in a plane is composed of horizontal line and a vertical line set perpendicular to each other in a plane. The lines are called the *coordinate axes*; their point of intersection, the *origin*. The horizontal line is generally referred to as the x-*axis*; the vertical line, the y-*axis*. The four sections in to which the plane is divided by the intersection of the axes are called *quadrants.*

Each point in the plane is uniquely associated with an ordered pair of numbers, known as *coordinates*, describing the location of the point in relation to the origin. The first coordinate, called the x-*coordinate* or *abscissa*, gives the distance of the point from the vertical axis; the second, the y- *coordinate* or *ordinate*, gives the distance of the point from the horizontal axis. To the right of the y-axis, x coordinates are positive; to the left, negative. Above the x-axis, y coordinates are positive; below, negative. The signs of the coordinates in each of the quadrants are illustrated in the following Figure.

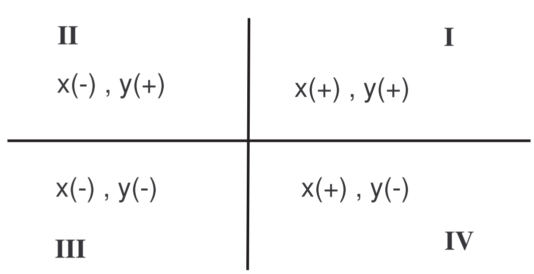

Figure 1.1

Example 1.2. The coordinates give the location of the point P in relation to the origin. The point $(4, 3)$ is four units to the right of the y- axis, three units above the x- axis. $P(-4, 3)$ is four units to the left of the y- axis, three units above the x- axis. $P(-4, -3)$ is four units to the left of the y- axis, three units below the x- axis; $P(4, -3)$ is four units to the right of the y- axis, three units below the x- axis. See Figure 1.2.

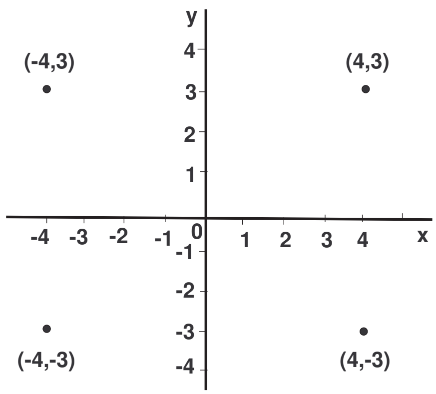

Figure 1.2

1.3 Graphing Linear Equations

For any equation in x and y, one can isolate a set of points in the Cartesian coordinate system. The set of points is called the *graph of the equation.* The graph of a linear equation is a *straight line.*

The *x-intercept* is the point where the graph crosses the x- axis ; the *y-intercept* is the point where the graph crosses the y- axis. Since the line crosses the x- axis where $y = 0$, the y-coordinate of the x-intercept is always 0. The x- coordinate of the x-intercept is then obtained simply by setting y equal to zero and solving the equation for x. Similarly the line crosses the y- axis where $x = 0$, the x-coordinate of the y-intercept is always 0. The y- coordinate of the y-intercept is then obtained simply by setting x equal to zero and solving the equation for y.

The ***slope-intercept form*** of a line is given by

$$y = mx + b \qquad (m, b \ = \ constants)$$

where m is the slope of the line and $(0, b)$ is the y- intercept. The y-coordinate of the y-intercept is always a constant in the slope-intercept form of the equation.

Problem 1.5. Put the following linear equations in to the slope-intercept form by solving for y in terms of x and /or a constant:

(a) $40x + 8y = 96$

(b) $18x - 9y = 27$

(c) $63x - 7y = 0$

Solution.

(a)

$$\begin{aligned} 40x + 8y &= 96 \\ 8y &= 96 - 40x \\ 8y &= -40x + 96 \\ y &= -5x + 12 \end{aligned}$$

(b)

$$\begin{aligned} 18x - 9y &= 27 \\ -9y &= 27 - 18x \\ -9y &= -18x + 27 \\ y &= -2x + 3 \end{aligned}$$

(c)

$$\begin{aligned} 63x - 7y &= 0 \\ -7y &= -63x \\ y &= 9x \end{aligned}$$

■

Problem 1.6. Find the y-intercept for each of the following equations :

(a) $3x + y = 7$

(b) $9x - 3y = 72$

(c) $y = 24x - 45$

Solution.

(a) The y-intercept occurs where the line crosses the y-axis, which is the point where $x = 0$. Setting $x = 0$ in the equation $3x + y = 7$ and solving for y we have,

$$\begin{aligned} 3(0) + y &= 7 \\ y &= 7 \end{aligned}$$

Hence the y-intercept is $(0, 7)$

(b) Setting $x = 0$,

$$\begin{aligned} 9(0) - 3y &= 72 \\ -3y &= 72 \\ y &= -24 \end{aligned}$$

Hence the y-intercept is $(0, -24)$

(c) Here the equation is in the slope-intercept form. Hence the y-intercept is $(0, -45)$.

■

Problem 1.7. Find the x-intercept for each of the following equations :

(a) $y = 6x - 54$

(b) $y = 12x + 132$

Solution.

(a) The x-intercept is the point where the line crosses the x-axis, which is the point where $y = 0$. Setting $y = 0$ in the equation $y = 6x - 54$ and solving for x, we have

$$\begin{aligned} 0 &= 6x - 54 \\ -6x &= -54 \\ x &= 9 \end{aligned}$$

Hence the x-intercept is $(9, 0)$

(b) Setting $y = 0$,

$$\begin{aligned} 0 &= 12x + 132 \\ -12x &= 132 \\ x &= -11 \end{aligned}$$

Hence the x-intercept is $(-11, 0)$.

■

Problem 1.8. Find the x-intercept in terms of the parameters of the slope-intercept form of a linear eqauation $y = mx + b$

Solution. Setting $y = 0$,

$$\begin{aligned} 0 &= mx + b \\ -mx &= b \\ x &= -\frac{b}{m} \qquad (1.1) \end{aligned}$$

Hence the x-intercept of the slope intercept form is $(-b/m, 0)$ ■

Problem 1.9. Use the information in Problem 1.8 to speed the process of finding the x-intercepts for the following equations :

(a) $y = 15x + 75$

(b) $y = 25x + 225$

Solution.

(a) Here $m = 15,\ b = 75$. Substituting in (1.1),

$$x = -\frac{75}{15} = -5$$

So the x-intercept is $(-5, 0)$.

(b) Here $m = 25,\ b = 225$. Substituting in (1.1),

$$x = -\frac{225}{25} = -9$$

So the x-intercept is $(-9, 0)$. ■

Problem 1.10. Find the y-intercepts and x-intercepts and use them as the two points needed to graph the following linear equations:

(a) $y = -4x + 8$

(b) $y = 2x + 4$

(c) $y = 2$

Solution.

(a) y-intercept is $(0, 8)$ and x-intercept is $(2, 0)$

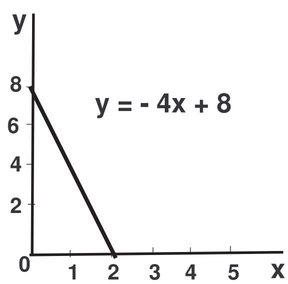

Figure 1.3

(b) y-intercept is $(0, 4)$ and x-intercept is $(-2, 0)$

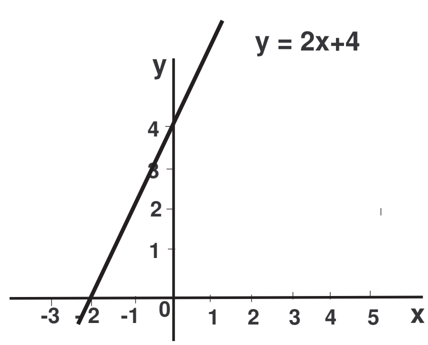

Figure 1.4

(c) y-intercept is $(0, 2)$, x-intercept does not exist because y cannot be set equal to zero without involving contradiction. Since $y = 2$ independently of x, y will equal to 2 for any value of x.

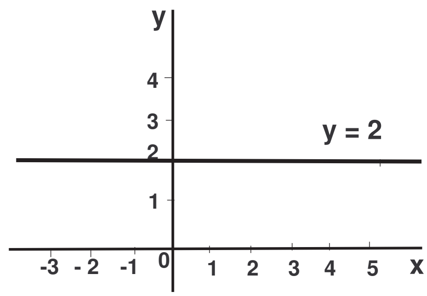

Figure 1.5

■

1.4 Solving Linear Equations Simultaneously

Two equations involving the same two variables can be solved algebraically or graphically. To solve algebraically (1) express the equations in the slope intercept form, (2) equate the two expressions for y and solve for x, and then (3) substitute the solution for x in either of the equations to find y. To solve graphically, graph the two equations on the same plane and look for the point of intersection . The coordinates of the point of intersection, represents the unique point common to both equations, provide the solution.

Problem 1.11. Solve each of the following systems of equations algebraically.

(a) $-5x + y = -8; \quad 6x - y = 11$

(b) $6x + 2y = 16; \quad -4x + y = -6$

Solution.

(a) (1) Setting each of the equations in slope-intercept form,

$$\begin{aligned} y &= 5x - 8 \\ y &= 6x - 11 \end{aligned}$$

(2) Equating y's and solving for x,

$$\begin{aligned} 5x - 8 &= 6x - 11 \\ 5x - 6x &= -11 + 8 \\ -x &= -3 \\ x &= 3 \end{aligned}$$

(3) Substituting $x = 3$ in either of the equations,

$$\begin{aligned} y &= 5(3) - 8 \\ y &= 15 - 8 \\ y &= 7 \end{aligned}$$

Hence the solution is $x = 3, \ \ y = 7$.

(b) (1) Setting each of the equations in slope-intercept form,

$$\begin{aligned} y &= -3x + 8 \\ y &= 4x - 6 \end{aligned}$$

(2) Equating y's and solving for x,

$$\begin{aligned} -3x + 8 &= 4x - 6 \\ -3x - 4x &= -6 - 8 \\ -7x &= -14 \\ x &= 2 \end{aligned}$$

(3) Substituting $x = 2$ in either of the equations,

$$\begin{aligned} y &= -3(2) + 8 \\ y &= -6 + 8 \\ y &= 2 \end{aligned}$$

Hence the solution is $x = 2, \ \ y = 2$. ■

Problem 1.12. Solve each of the following system of equations graphically:

(a) $y = -2x + 10; \qquad y = \frac{1}{4}x + 1$

(b) $y = -\frac{1}{5}x + 3; \qquad y = 2x - 8$

Solution.

(a) For the line $y = -2x + 10$, x-intercept is $(5, 0)$ and y- intercept is $(0, 10)$. For $y = \frac{1}{4}x + 1$, x-intercept is $(-4, 0)$ and y- intercept is $(0, 1)$.

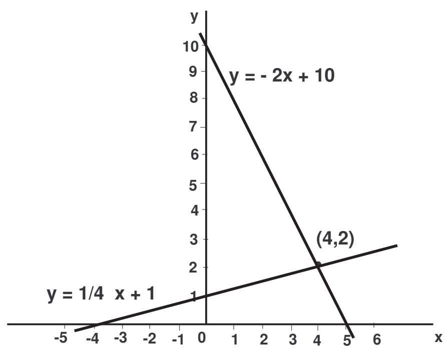

Figure 1.6

From Figure the point of intersection is $(4, 2)$.So the solution is $x = 4, y = 2$.

(b) Intercepts : $(0, 3), (15, 0); (0, -8), (4, 0)$

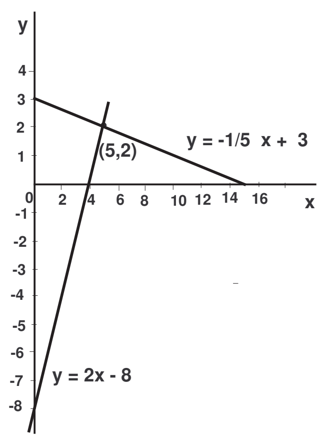

Figure 1.7

From Figure the point of intersection is $(5, 2)$. So the solution is $x = 5, y = 2$.

1.5 Slope of a Straight Line

The slope indicates the steepness and direction of a line. The slope of a horizontal line $y = k$ (a constant) is zero. The slope of a vertical line $x = a$ (a constant) is undefined.

For a line passing through the points (x_1, y_1) and (x_2, y_2), the slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1 - y_2}{x_1 - x_2} \quad (x_1 \neq x_2)$$

A line with slope m and passing through the point (x_1, y_1) has the equation, called the ***point-slope form***

$$y - y_1 = m(x - x_1)$$

Problem 1.13. Find the slope of the lines represented by the following lines:

(a) $y = -7x + 13$

(b) $y = 5x + 3$

(c) $15x + 5y = 40$

(d) $8x - 2y = 28$

Solution.

(a) In the slope-intercept form of a linear equation, the coefficient of x is the slope of the line. The given equation is in slope-intercept form, so its slope is -7.

(b) The given equation is in slope-intercept form, so its slope is coefficient of x which is equal to 5.

(c) First convert the given equation in to slope-intercept form,

$$\begin{aligned} 15x + 5y &= 40 \\ 5y &= 40 - 15x \\ 5y &= -15x + 40 \\ y &= -3x + 8 \end{aligned}$$

The slope intercept form of given equation is $y = -3x + 8$, so its slope is -3.

(d) First convert the given equation in to slope-intercept form,

$$\begin{aligned} 8x - 2y &= 28 \\ -2y &= 28 - 8x \\ -2y &= -8x + 28 \\ y &= 4x - 14 \end{aligned}$$

The slope intercept form of given equation is $y = 4x - 14$, so its slope is 4.

■

Problem 1.14. Find the slope of the line passing through the points $(2, 3)$ and $(5, 12)$.

Solution. Let $(x_1, y_1) = (2, 3)$ and $(x_2, y_2) = (5, 12)$. The slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{12 - 3}{5 - 2} = \frac{9}{3} = 3.$$

So the slope of the line passing through the points $(2, 3)$ and $(5, 12)$ is 3. ■

Problem 1.15. Find the slope of the line passing through the points $(1, 7)$ and $(5, 15)$.

Solution. Let $(x_1, y_1) = (1, 7)$ and $(x_2, y_2) = (5, 15)$. The slope

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{15 - 7}{5 - 1} = \frac{8}{4} = 2.$$

So the slope of the line passing through the points $(1, 7)$ and $(5, 15)$ is 2. ■

Problem 1.16. Find the equation of the line passes through the point $(3, 10)$ and has slope -4.

Solution. Let $(x_1, y_1) = (3, 10)$, given slope $m = -4$. Then equation of the line is,

$$y - y_1 = m(x - x_1)$$

$$\begin{aligned} y - 10 &= -4(x-3) \\ y - 10 &= -4x + 12 \\ y &= -4x + 12 + 10 \\ y &= -4x + 22 \end{aligned}$$

■

Problem 1.17. Find the equation of the line passes through the point $(-8, 2)$ and has slope 3.

Solution. Let $(x_1, y_1) = (-8, 2)$, given slope $m = 3$. Then the equation of the line is,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 2 &= 3(x - (-8)) \\ y - 2 &= 3(x + 8) \\ y - 2 &= 3x + 24 \\ y &= 3x + 24 + 2 \\ y &= 3x + 26 \end{aligned}$$

■

Problem 1.18. Find the equation for the line passing through $(-2, 5)$ and parallel to the line having the equation $y = 3x + 7$.

Solution. *Parallel lines* have the same slope . The slope of the line $y = 3x + 7$ is 3. So we have to find the equation for the line passing through $(-2, 5)$ with slope 3. Let $(x_1, y_1) = (-2, 5)$, here slope $m = 3$. Then the equation of the line is,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 5 &= 3(x - (-2)) \\ y - 5 &= 3x + 6 \\ y &= 3x + 6 + 5 \\ y &= 3x + 11 \end{aligned}$$

■

Problem 1.19. Find the equation for the line passing through $(8,3)$ and perpendicular to the line having the equation $y = 4x + 13$.

Solution. *Perpendicular lines* have slopes that are negative reciprocals of one another. The slope of the line $y = 4x + 13$ is 4, the slope of a line perpendicular to it must be $-\frac{1}{4}$. So we have to find the equation for the line passing through $(8,3)$ with slope $-\frac{1}{4}$. Let $(x_1, y_1) = (8,3)$, here slope $m = -\frac{1}{4}$. Then the equation of the line is,

$$\begin{aligned} y - y_1 &= m(x - x_1) \\ y - 3 &= -\frac{1}{4}(x - 8) \\ y - 3 &= -\frac{1}{4}x + 2 \\ y &= -\frac{1}{4}x + 2 + 3 \\ y &= -\frac{1}{4}x + 5 \end{aligned}$$

■

1.6 Solving Quadratic Equations

An equation of the form $ax^2 + bx + c = 0$ where a, b, and c are constants and $a \neq 0$ is called a *quadratic equation.* Quadratic equations can be solved by factoring, *completing the square* or using the *quadratic formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{1.2}$$

Factoring is the reverse process of multiplication by which a polynomial is expressed as a product of simpler polynomials called factors.

Note 1.3. Let p and q be two integers such that $p = a + b$ and $q = ab$ for some integers a, b. Then we can write $x^2 + px + q$ as $(x + a)(x + b)$.

Note 1.4. An expression in the form $x^2 + bx$ can be converted in to perfect square by adding the square of the half of the coefficient of x, that is $\left(\frac{b}{2}\right)^2$ to the original expression. Thus we get $x^2 + bx + \left(\frac{b}{2}\right)^2 = \left(x + \frac{b}{2}\right)^2$

Problem 1.20. Solve the following quadratic equations by factoring:

(a) $x^2 + 12x + 35 = 0$

(b) $x^2 - 4x - 32 = 0$

Solution.

(a) Comparing the given equation with $x^2 + px + q$ we get $p = 12$ and $q = 35$, that is, $a + b = 12$ and $ab = 35$. The numbers that satisfies above two equations are 7 and 5. So,

$$x^2 + 12x + 35 = (x + 5)(x + 7) = 0$$

$$\Rightarrow (x + 5) = 0 \; or \; (x + 7) = 0$$

$$\Rightarrow x = -5 \; or \; x = -7$$

So solution of $x^2 + 12x + 35 = 0$ is $x = -5, -7$.

(b) Comparing the given equation with $x^2 + px + q$ we get $p = -4$ and $q = -32$, that is, $a + b = -4$ and $ab = -32$. The numbers that satisfies above two equations are -8 and 4. So,

$$x^2 - 4x - 32 = (x + 4)(x - 8) = 0$$

$$\Rightarrow (x + 4) = 0 \; or \; (x - 8) = 0$$

$$\Rightarrow x = -4 \; or \; x = 8$$

So solution of $x^2 - 4x - 32 = 0$ is $x = -4, 8$.

■

Problem 1.21. Solve the following quadratic equations using the quadratic formula:

(a) $3x^2 + 20x + 12 = 0$

(b) $11x^2 + x - 12 = 0$

Solution. Using 1.2 and substituting $a = 3, b = 20$, and $c = 12$,

$$\begin{aligned} x &= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \\ &= \frac{-20 \pm \sqrt{(20)^2 - 4(3)(12)}}{2(3)} \\ &= \frac{-20 \pm \sqrt{400 - 144}}{6} \\ &= \frac{-20 \pm \sqrt{256}}{6} \\ &= \frac{-20 \pm 16}{6} \end{aligned}$$

$$x = \frac{-20 + 16}{6} = \frac{-4}{6} = \frac{-2}{3}\ , \quad x = \frac{-20 - 16}{6} = \frac{-36}{6} = -6.$$

■

Problem 1.22. Complete the square and write the following expressions as perfect squares :

(a) $x^2 + 6x$

(b) $x^2 - 14x$

(c) $x^2 - \frac{6}{5}$

Solution.

(a) Comparing given equation with the equation $x^2 + bx$, we get $b = 6$, $\frac{b}{2} = \frac{6}{2} = 3$ and $\left(\frac{b}{2}\right)^2 = 3^2 = 9$. Adding 9 to the original expression and factoring,

$$x^2 + 6x + 9 = (x + 3)^2$$

(b) Comparing given equation with the equation $x^2 + bx$, we get $b = -14$, $\frac{b}{2} = \frac{-14}{2} = -7$ and $\left(\frac{b}{2}\right)^2 = (-7)^2 = 49$. Adding 49 to the original expression and factoring,

$$x^2 - 14x + 49 = (x - 7)^2$$

(c) Comparing given equation with the equation $x^2 + bx$, we get $b = \frac{-6}{5}$, $\frac{b}{2} = \frac{-6}{5 \cdot 2} = \frac{-3}{5}$ and $\left(\frac{b}{2}\right)^2 = \left(\frac{-3}{5}\right)^2 = \frac{9}{25}$. Adding $\frac{9}{25}$ to the original expression and factoring,

$$x^2 - \frac{6}{5}x + \frac{9}{25} = \left(x - \frac{3}{5}\right)^2$$

■

Problem 1.23. Use the process of completing the square to solve the following quadratic equations:

(a) $x^2 + 12x + 32 = 0$

(b) $x^2 - 10x + 13 = 0$

(c) $-x^2 + 8x + 20 = 0$

(d) $3x^2 + 24x + 30 = 0$

Solution.

(a) Move the constant term to the right-hand side we get,

$$x^2 + 12x = -32$$

Ignore the constant for the moment and complete the square on the left as in Problem 1.22, getting $b = 12$, $\frac{12}{2} = 6$ and $6^2 = 36$. Now add 36 to both sides of the equation to keep the equality,

$$x^2 + 12x + 36 = -32 + 36$$

Next factor the left-hand side to obtain the perfect square,

$$(x + 6)^2 = 4$$

Take the square root of both sides of the equation and solve for x,

$$\begin{aligned} x + 6 &= \pm\sqrt{4} \\ x &= -6 \pm \sqrt{4} \\ x &= -6 \pm 2 \end{aligned}$$

$$\Rightarrow x = -6 + 2 = -4, \qquad x = -6 - 2 = -8.$$

(b) Moving the constant term to the right-hand side we get,

$$x^2 - 10x = -13$$

Here $b = -10$, $\frac{-10}{2} = -5$ and $(-5)^2 = 25$. Now add 25 to both sides of the equation to keep the equality,

$$x^2 - 10x + 25 = -13 + 25$$

Next factor the left-hand side to obtain the perfect square,

$$(x - 5)^2 = 12$$

Take the square root of both sides of the equation and solve for x,

$$\begin{aligned} x - 5 &= \pm\sqrt{12} \\ x &= 5 \pm \sqrt{12} \\ x &= 5 \pm 2\sqrt{3} \end{aligned}$$

$$\Rightarrow x = 5 + 2\sqrt{3}, \qquad x = 5 - 2\sqrt{3}$$

(c) Here coefficient of x^2 is -1, so we first factoring out -1. Then our equation become

$$x^2 - 8x - 20 = 0$$

Moving the constant term to the right-hand side we get,

$$x^2 - 8x = 20$$

Here $b = -8$, $\frac{-8}{2} = -4$ and $(-4)^2 = 16$. Adding 16 and solving,

$$\begin{aligned} x^2 - 8x + 16 &= 20 + 16 \\ (x - 4)^2 &= 36 \\ x - 4 &= \pm\sqrt{36} \\ x &= 4 \pm \sqrt{36} \\ x &= 4 \pm 6 \end{aligned}$$

$$\Rightarrow x = 4 + 6 = 10, \qquad x = 4 - 6 = -2.$$

(d) Here coefficient of x^2 is 3, factoring out 3. Then our equation become

$$x^2 + 8x + 10 = 0$$

Moving the constant term to the right-hand side we get,

$$x^2 + 8x = -10$$

Here $b = 8$, $\frac{8}{2} = 4$ and $(4)^2 = 16$. Adding 16 and solving,

$$\begin{aligned} x^2 + 8x + 16 &= -10 + 16 \\ (x+4)^2 &= 6 \\ x + 4 &= \pm\sqrt{6} \\ x &= -4 \pm \sqrt{6} \end{aligned}$$

$$\Rightarrow x = -4 + \sqrt{6}, \qquad x = -4 - \sqrt{6}$$

■

1.7 Practical Applications of Graphs and Equations

Problem 1.24. A firm has a fixed cost of \$ 4,000 for plant and equipment and an extra or marginal cost of \$ 300 for each additional unit produced. What is its total cost, C, of producing

(a) 25 units of output.

(b) 40 units of output.

Solution. Fixed cost is \$4,000 and cost for producing one additional unit is \$300. So total cost for producing x unit is

$$C = 300x + 4,000$$

(a) If $x = 25$, $\quad C = 300(25) + 4,000 = 11,500$

(b) If $x = 40$, $\quad C = 300(40) + 4,000 = 16,000$

Problem 1.25. A firm operating in pure competition receives \$25 for each unit of output sold. It has a marginal cost of \$15 per item and a fixed cost of \$1,200. What is its profit level, π, if it sells

(a) 200 items.

(b) 300 items.

(c) 100 items.

Solution.

$$profit(\pi) = revenue(R) - cost(C)$$

Here revenue $R = 25x$ and $C = 15x + 1,200$
Substituting we get,

$$\pi = 25x - (15x + 1,200) = 10x - 1,200$$

(a) at $x = 200$, $\quad \pi = 10(200) - 1,200 = 800$

(b) at $x = 300$, $\quad \pi = 10(300) - 1,200 = 1,800$

(c) at $x = 100$, $\quad \pi = 10(100) - 1,200 = -200$ (a loss)

■

Problem 1.26. For tax purpose the value y of a factory after x years is

$$y = 9,000,000 - 850,000x$$

(a) The value after 4 years.

(b) The salvage value after 9 years.

Solution.

(a) $\quad y = 9,000,000 - 850,000(4) = 5,600,000$

(b) $\quad y = 9,000,000 - 850,000(9) = 1,350,000$ ■

Problem 1.27. Find the break-even point for the firm operating in pure competition, given that total revenue is $R = 25x$ and total cost is $C = 15x + 1200$.

Note 1.5. At the break-even point total revenue just equals total cost. So at break-even point profit is zero.

At break-even point,

$$\begin{aligned} R &= C \\ 25x &= 15x + 1200 \\ 25x - 15x &= 1200 \\ 10x &= 1200 \\ x &= 120 \end{aligned}$$

Problem 1.28. Find the break-even for a firm operating on monopolistic competition, given that total revenue is $R = 48x - x^2$ and total cost is $TC = 6x + 120$.

The break-even occurs when

$$\begin{aligned} R &= TC \\ 48x - x^2 &= 6x + 120 \\ 0 &= 6x + 120 - 48x + x^2 \\ 0 &= 120 - 42x + x^2 \\ \Rightarrow \quad x^2 - 42x + 120 &= 0 \\ \Rightarrow \quad x &= \frac{-(-42) \pm \sqrt{(-42)^2 - 4(1)(120)}}{2(1)} \\ &= \frac{42 \pm \sqrt{1764 - 480}}{2} \\ &= \frac{42 \pm \sqrt{1284}}{2} \\ &= \frac{42 \pm 2 \cdot \sqrt{321}}{2} \\ &= 21 \pm \sqrt{321} \end{aligned}$$

$$\Rightarrow \qquad x = 21 + \sqrt{321}, \qquad x = 21 - \sqrt{321}$$

So break- even points are $x = 21 + \sqrt{321}$, $\quad x = 21 - \sqrt{321}$

Problem 1.29. Find the equilibrium price p_0 and quantity q_0 given Supply : $q = 30p - 280$ and Demand : $q = -16p + 410$.

Note 1.6. Equilibrium price is the market price at which the supply of an item equals the quantity demanded.

Equating supply and demand for equilibrium and solving for p,

$$\begin{aligned} 30p - 280 &= -16p + 410 \\ 30p + 16p &= 410 + 280 \\ 46p &= 690 \\ p &= \frac{690}{46} = 15 \end{aligned}$$

So equilibrium price $p_0 = 15$. Substitute p_0 either in supply or in demand to find q_0.

$$q_0 = 30(15) - 280 = 170.$$

1.8 Exercises

1. Solve the following equations

   (a) $2x + 5 = 5x - 4$

   (b) $10x - 45 = 5(2x - 12) + 15$

   (c) $\frac{x}{3} - 16 = \frac{x}{12} + 14$

   (d) $\frac{48}{x-5} - \frac{45}{x} = \frac{28}{x-5} \quad (x \neq 5)$

2. In the following equations solve for x by clearing the denominator, that is, by multiplying both sides of the equation by the least common denominator as soon as is feasible.

   (a) $\frac{x}{3} - 16 = \frac{x}{12} + 14$

   (b) $\frac{5}{x} + \frac{3}{x+4} = \frac{7}{x} \quad (x \neq 0, -4)$

   (c) $\frac{48}{x-5} - \frac{45}{x} = \frac{28}{x-5} \quad (x \neq 0, 5)$

3. Solve each of the following systems of equations algebraically:

   (a) $\frac{1}{2}x + y = 7; \quad x + 3y = 15$

   (b) $7x - y = 3; \quad 5x + y = 2$

   (c) $-8x + 2y = 10; \quad \frac{2}{3}x - \frac{1}{3}y = 1$

4. Find the y-intercepts and x-intercepts and use them as the two points needed to graph the following linear equations:

   (a) $y = 3x - 9$

   (b) $y = \frac{1}{4}x - 2$

   (c) $y = 3x$

5. Solve each of the following systems of equations graphically.

   (a) $y = -2x + 10; \quad y = 6x + 2$

   (b) $2x - y = 8; \quad 9x + 3y = 21$

   (c) $5x + y = 15; \quad 2x - y = -1$

6. Find the slope of the lines represented by the following lines

   (a) $y = \frac{1}{3}x - 5$

   (b) $y = -9x$

   (c) $y = 15$

   (d) $x = 4$

   (e) $12y - 30x = 60$

   (f) $18x - 8y = 4$

7. Find the slope of the line passing through the following points.

   (a) $(3, 11)$ and $(6, 2)$

   (b) $(-9, -14)$ and $(-5, -4)$

8. Write an equation for each of the lines below:

   (a) Line passing through $(-8, 2)$ with slope 3

   (b) Line passing through $(6, -4)$ with slope $\frac{1}{2}$

9. Solve the following quadratic equations by factoring:
   (a) $x^2 + 13x + 40 = 0$
   (b) $x^2 + 31x - 66 = 0$
10. Solve the following quadratic equations using the quadratic formula:
    (a) $5x^2 + 31x + 30 = 0$
    (b) $7x^2 - 28x + 28 = 0$
11. Use the process of completing the square to solve the following quadratic equations:
    (a) $x^2 - 6x - 27 = 0$
    (b) $x^2 - 18x + 76 = 0$
    (c) $-5x^2 - 30x + 25 = 0$
    (d) $11x^2 + x - 12 = 0$
12. A firm has a fixed cost of $1,500 for plant and equipment and an extra or marginal cost of $200 for each additional unit produced. What is its total cost, C, of producing
    (a) 15 units of output.
    (b) 20 units of output.
13. Find the profit level of a firm in pure competition that has fixed cost of $750, a marginal cost of $80, and selling price of $95 when it sells
    (a) 40 units.
    (b) 60 units.
14. Find the equilibrium price p_0 and quantity q_0 given Supply : $q = 200p - 1400$ and Demand : $q = -50p + 1850$.

Chapter 2

Functions

2.1 Concepts and Definitions

A *function* from a set A in to a set B is a rule that assigns each elements in a set A to *exactly one* element in a set B. The set A is called the *domain* of a function (the set of input) and the set B is called the *range* of the function. (the set of output) Usually we will denote it by $y = f(x)$, where x is the input and is called the *independent variable* and y is the output and is called the *dependent variable.* $f(x)$ we read as "f of x"or the "value of f at x". The *domain* of a function refers to the set of all possible values for the independent variable; the *range* refers to the set of all possible values for the dependent variable.

Example 2.1. The function

$$f(x) = \frac{x}{2} + 7$$

is the rule that takes a number, divides it by 2, and then adds 7 to the quotient. If a value is given for x, the value is substituted for x in the formula and the equation solved for $f(x)$. For example, if $x = 4$, $f(4) = \frac{4}{2} + 7 = 9$. If $x = 6$, $f(6) = \frac{6}{2} + 7 = 10$.

Example 2.2. If x represents the speed limit in miles per hour, then the speed limit in kilometers per hour is a function of x, represented by $f(x) = 1.6094x$. If the speed limit in the United States is 55 mi/h, its kilometre equivalent, when rounded to the nearest integer is,

$$f(55) = 1.6094(55) = 89 \; km/h$$

Some important functions

Linear function: $f(x) = mx + b$
Quadratic function : $f(x) = ax^2 + bx + c \ \ (a \neq 0)$
Polynomial function of degree n :

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \ \ (n = non\ negative\ integer;\ a_n \neq 0)$$

Rational function: $f(x) = \dfrac{g(x)}{h(x)}$
where $g(x)$ and $h(x)$ are both polynomials and $h(x) \neq 0$
Power function : $f(x) = ax^n \qquad (n = \text{any real number})$

The domain of linear, quadratic, and polynomial functions is the set of all real numbers; the domain of rational and power functions excludes any value of x involving an undefined operation.

Example 2.3. Listed below are examples of different functions:

Linear : $f(x) = 3x + 2, \qquad g(x) = -5x, \qquad h(x) = 8$
Quadratic : $f(x) = 5x^2 + 4x + 3, \qquad g(x) = x^2 - 4x, \qquad h(x) = 4x^2$
Polynomial : $f(x) = 4x^3 + 2x^2 - 9x + 5, \qquad g(x) = 2x^5 - x^3 + 7$
Rational : $f(x) = \dfrac{x^2 - 9}{x + 4}, \qquad g(x) = \dfrac{5x}{x - 2} \qquad (x \neq -4, 2)$
Power: $f(x) = 2x^6, \qquad g(x) = 4x^{-3}$

Problem 2.1. .

(a) Given $f(x) = x^2 + 5x - 6$, find $f(3)$ and $f(-4)$.

(b) Given $f(x) = \dfrac{4x^2 - 9x + 17}{x + 7}$, find $f(5)$ and $f(-3)$.

(c) Given $f(x) = \dfrac{x^2 - 11}{x + 4}$, find $f(a)$ and $f(a - 5)$.

Solution.

(a) Substituting 3 for each occurrence of x in the function, we have

$$f(3) = (3)^2 + 5(3) - 6 = 18$$

Now substituting -4 for each occurrence of x,

$$f(-4) = (-4)^2 + 5(-4) - 6 = -10$$

(b) Substituting 5 for each occurrence of x in the function, we have

$$f(5) = \frac{4(5)^2 - 9(5) + 17}{5 + 7} = \frac{72}{12} = 6$$

Now substituting -3 for each occurrence of x,

$$f(-3) = \frac{4(-3)^2 - 9(-3) + 17}{(-3) + 7} = \frac{80}{4} = 20$$

(c) Substituting a for each occurrence of x in the function, we have

$$f(a) = \frac{a^2 - 11}{a + 4}$$

Now substituting $(a - 5)$ for each occurrence of x,

$$f(a-5) = \frac{(a-5)^2 - 11}{(a-5) + 4} = \frac{a^2 - 10a + 14}{a - 1}$$

■

Problem 2.2. Which of the following equations are functions and why?

(a) $y = -2x + 7$

(b) $y^2 = x$

(c) $x = 4$

(d) $-x^2 + 6x + 15$

(e) $y = x^2$

(f) $x^2 + y^2 = 36$

Solution.

(a) $y = -2x+7$ is a function because for each value of the independent variable x there is one and only one value of the dependent variable y.

(b) $y^2 = x$, which is equivalent to $y = \pm\sqrt{x}$ is not a function because for each positive value of x there are two values of y. For example, if $y^2 = 4$ then $y = \pm 2$.

(c) $x = 4$ is not a function. The graph of $x = 4$ is vertical line. This means that at $x = 4$, y has many values.

(d) $y = -x^2 + 6x + 15$ is a function because for each value of the variable x there is a unique value of y.

(e) $y = x^2$ is a function. Because for each value of the variable x there is a unique value of y.

(f) $x^2 + y^2 = 36$ is not a function. If $x = 0$, $y^2 = 36$ and $y = \pm 6$.

■

Problem 2.3. Identify the domain of the following functions:

(a) $y = 4x^2 + 7x - 19$

(b) $y = \sqrt{t - 5}$

(c) $y = \dfrac{7}{x(x-4)}$

(d) $y = \dfrac{6x}{(x-5)(x-9)}$

Solution.

(a) The domain of a function is the set of all acceptable values of the independent variable. Since x may assume any value in (a), the domain of the function is the set of all real numbers.

(b) A square root is defined only for non negative numbers (that is , $x \geq 0$), it is necessary that $t - 5 \geq 0$, that is $t \geq 5$. So domain of the function is $\{t : t \geq 5\}$.

(c) Because division by zero is not permissible, $x(x-4)$ cannot equal zero. The domain of the function excludes $x = 0$ and $x = 4$. So the domain of the function is $\{x : x \neq 0, 4\}$.

(d) Because division by zero is not permissible, $(x-5)(x-9)$ cannot equal zero. The domain of the function excludes $x = 5$ and $x = 9$. So the domain of the function is $\{x : x \neq 5, 9\}$.

■

Problem 2.4. Identify the range of the following functions:

(a) $y = 2x$

(b) $y = x^2$

(c) $y = 3x - 2 \quad (-2 \leq x \leq 2)$

Solution.

(a) The range of a function is the set of all possible values of the dependent variable y. Here the range is the set of all real numbers since the domain includes all real numbers.

(b) Since the square of a number cannot be negative, the range includes all non negative numbers (that is $y \geq 0$).

(c) The range of y is $(-8 \leq y \leq 4)$.

■

2.2 Functions and Graphs

The graph of a linear function is a straight line.To graph a nonlinear function, simply pick some representative value of x; solve for $f(x)$, which is usually referred to as y in graphing; plot the resulting ordered pairs $(x, f(x))$; and connect them with a smooth line.

Problem 2.5. Plot the graph of $y = x^2$.

Solution.

x	-3	-2	-1	0	1	2	3
$f(x) = x^2$	9	4	1	0	1	4	9

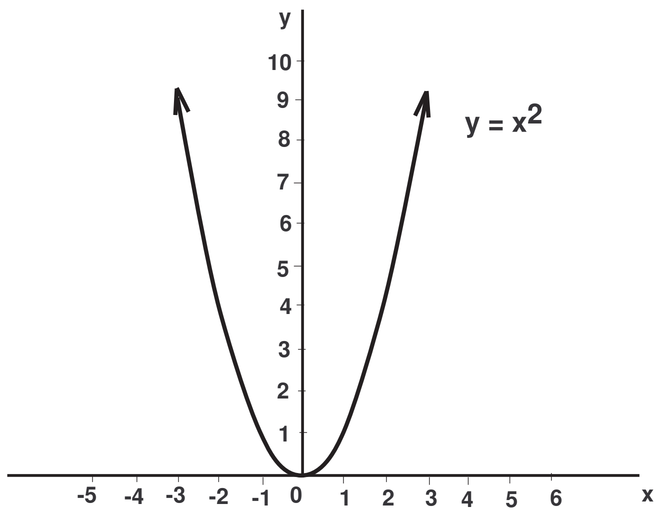

Figure 2.1

■

Problem 2.6. Draw the graph of $y = \dfrac{1}{x}$.

Solution.

x	-4	-2	-1	1	2	4
$f(x) = 1/x$	-0.25	-0.5	-1	1	0.5	0.25

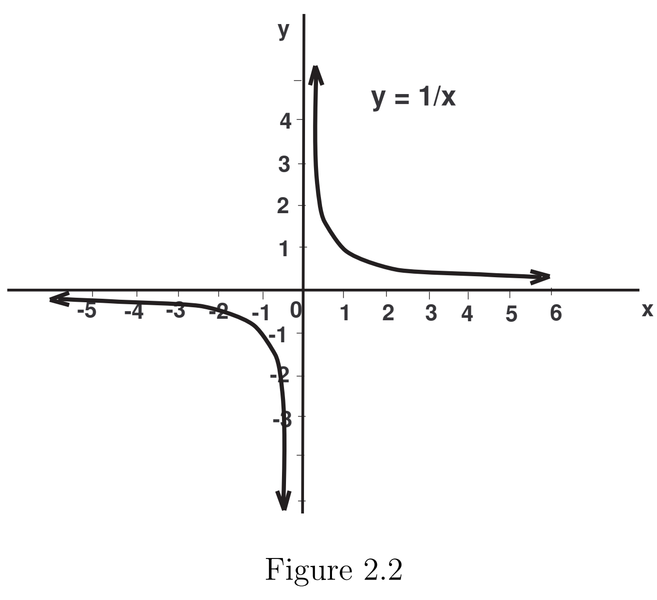

Figure 2.2

■

The graph of a *quadratic function* of the form $f(x) = ax^2 + bx + c$, where $a \neq 0$, is a *parabola*. The graph of a parabola is symmetric about a line called the *axis of symmetry*. The point of intersection of the parabola and its axis is called the *vertex*. In Figure 2.1, the axis coincides with the y-axis; the vertex is $(0, 0)$.

2.3 The Algebra of Functions

Two or more functions can be combined to obtain a new function by addition, subtraction, multiplication, or division of the original functions. Given two functions f and g , with x in the domain of both f and g,

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
(f-g)(x) &= f(x)-g(x)\\
(f\cdot g)(x) &= f(x)\cdot g(x)\\
(f\div g)(x) &= f(x)\div g(x) \qquad [g(x)\neq 0]
\end{aligned}
$$

Functions can also be combined by substituting one function $f(x)$ for every occurrence of x in another function $g(x)$. This is known as a *composition of functions* and is denoted by $g[f(x)]$ or $(g \circ f)(x)$.

Problem 2.7. If $f(x) = 5x + 3$ and $g(x) = 4x - 8$, then find $(f + g)(x)$, $(f - g)(x), (f \cdot g)(x)$ and $(f \div g)(x)$.

Solution. We have,

$$
\begin{aligned}
(f+g)(x) &= f(x)+g(x)\\
(f+g)(x) &= (5x+3)+(4x-8) = 9x-5\\
(f-g)(x) &= f(x)-g(x)\\
(f-g)(x) &= (5x+3)-(4x-8) = x+11\\
(f\cdot g)(x) &= f(x)\cdot g(x)\\
(f\cdot g)(x) &= (5x+3)\cdot(4x-8) = 20x^2-28x-24\\
(f\div g)(x) &= f(x)\div g(x) \qquad [g(x)\neq 0]\\
(f\div g)(x) &= \frac{(5x+3)}{(4x-8)} \qquad (x\neq 2)
\end{aligned}
$$

■

Problem 2.8. If $f(x) = \dfrac{4}{x}$ and $g(x) = \dfrac{3}{x+2}$, then find $(f+g)(x), (f-g)(x)$, $(f \cdot g)(x)$ and $(f \div g)(x)$.

Solution.

$$\begin{aligned}
(f+g)(x) &= \frac{4}{x} + \frac{3}{x+2} = \frac{4(x+2)+3x}{x(x+2)} = \frac{4x+8+3x}{x^2+2x} = \frac{7x+8}{x^2+2x} \\
(f-g)(x) &= \frac{4}{x} - \frac{3}{x+2} = \frac{4(x+2)-3x}{x(x+2)} = \frac{4x+8-3x}{x^2+2x} = \frac{x+8}{x^2+2x} \\
(f \cdot g)(x) &= \frac{4}{x} \cdot \frac{3}{x+2} = \frac{12}{x(x+2)} = \frac{12}{x^2+2x} \\
(f \div g)(x) &= \frac{4}{x} \div \frac{3}{x+2} = \frac{4}{x} \cdot \frac{x+2}{3} = \frac{4(x+2)}{3x} = \frac{4x+8}{3x}
\end{aligned}$$

■

Problem 2.9. If $f(x) = \dfrac{3x}{x+5}$, $g(x) = \dfrac{x-4}{x+1}$ and $h(x) = \dfrac{x+6}{x-2}$, where $(x \neq -5, -1, 2)$ find :

(a) $(f+h)(a)$

(b) $(h-f)(x+1)$

Solution.

(a) $(f+h)(a) = f(a) + h(a)$. Substituting a for each occurrence of x,

$$\begin{aligned}
(f+h)(a) &= \frac{3a}{a+5} + \frac{a+6}{a-2} \\
&= \frac{3a(a-2)+(a+6)(a+5)}{(a+5)(a-2)} \\
&= \frac{3a^2-6a+a^2+6a+5a+30}{a^2+5a-2a-10} \\
&= \frac{4a^2+5a+30}{a^2+3a-10}
\end{aligned}$$

(b) $(h-f)(x+1) = h(x+1) - f(x+1)$. Substituting $x+1$ for each occurrence of x,

$$\begin{aligned}(h-f)(x+1) &= \frac{(x+1)+6}{(x+1)-2} - \frac{3(x+1)}{(x+1)+5}\\ &= \frac{x+7}{x-1} - \frac{3x+3}{x+6} \qquad (x \neq 1, -6)\\ &= \frac{(x+7)(x+6)-(3x+3)(x-1)}{(x-1)(x+6)}\\ &= \frac{(x^2+6x+7x+42)-(3x^2-3x+3x-3)}{x^2-x+6x-6}\\ &= \frac{-2x^2+13x+45}{x^2+5x-6}\end{aligned}$$

■

Problem 2.10. If $f(x) = x^3$, $g(x) = x^2 - 2x + 5$ and $h(x) = \frac{x}{x+4}$ where $(x \neq -4)$, find the following composite functions:

(a) $(g \circ f)(x)$

(b) $(f \circ h)(x)$

(c) $h[f(x)]$

Solution.

(a) $(g \circ f)(x) = g[f(x)]$, Substituting $f(x) = x^3$ for each occurrence of x in $g(x)$,

$$(g \circ f)(x) = g[f(x)] = (x^3)^2 - 2(x^3) + 5 = x^6 - 2x^3 + 5$$

(b) $(f \circ h)(x) = f[h(x)]$, Substituting $h(x)$ for each occurrence of x in $f(x)$,

$$(f \circ h)(x) = f[h(x)] = (h(x))^3 = \left(\frac{x}{x+4}\right)^3$$

(c) Substituting $f(x)$ for each occurrence of x in $h(x)$,

$$h[f(x)] = \frac{f(x)}{f(x)+4} = \frac{x^3}{x^3+4} \qquad (x^3 \neq -4 \; or \; x \neq \sqrt[3]{-4})$$ ■

2.4 Application of Linear Functions

Linear functions are used often in business, economics, and science, and frequently combined to form new functions. In economics we use $C(x)$ to represent a cost function, $R(x)$ to represent a revenue function, and $\pi(x)$ to represent a profit function.

Problem 2.11. An author receives a fee of \$ 5,000 plus \$ 3.50 for every book sold. Express his revenue R as a function of the number of books x sold.

Solution. Revenue $R(x) = 3.50x + 5,000.$ ■

Problem 2.12. The owner of a commercially stocked fishing pond charges \$10 to fish and \$0.50 a pound for whatever is caught. Express the cost of fishing C as a function of number of pounds of fish caught x.

Solution. Cost $C(x) = 0.50x + 10$ ■

Problem 2.13. A tool company sold 5,000 tool kits in 1980 and 20,000 tool kits in 1985. Assuming that sales are approximated by a linear function, express the company's sales S as a function of time t.

Solution. The general form of a linear function is $S(t) = mt + b$, where $m =$ the slope and $b =$ the vertical intercept. Letting $0 = 1980$ and $5 = 1985$,

$$m = \frac{20,000 - 5,000}{5 - 0} = \frac{15,000}{5} = 3,000$$

Given that in 1980 company sold 5,000 tool kits. Therefore $S = 5,000$ at $t = 0$. So,

$$5,000 = m(0) + b$$

$$\Rightarrow b = 5,000$$

Hence,

$$S(t) = 3,000t + 5,000.$$

■

2.5 Facilitating Nonlinear Graphs

By completing the square, quadratic functions can be expressed in the form

$$y = a(x - h)^2 + k \tag{2.1}$$

where the axis is $(x - h) = 0$, $x = h$; and the vertex is (h, k). If $a > 0$, the parabola opens up and the vertex is the *lowest point* of the function; if $a < 0$, the parabola opens down and the vertex is the *highest point*.

Problem 2.14. Find the vertex and axis of the parabola $y = -(x-3)^2 + 16$ and then draw the parabola.

Solution. Given

$$y = -(x - 3)^2 + 16$$

Comparing with the equation 2.1 we get $a = -1, h = 3, k = 16$. So axis is $x = h$ that is $x = 3$, vertex is $(h, k) = (3, 16)$, Here $a = -1 < 0$ so parabola is opens down. Thus we get graph of above quadratic function is a parabola with vertex at $(3, 16)$, symmetrical about the axis $x = 3$ and it opens down.

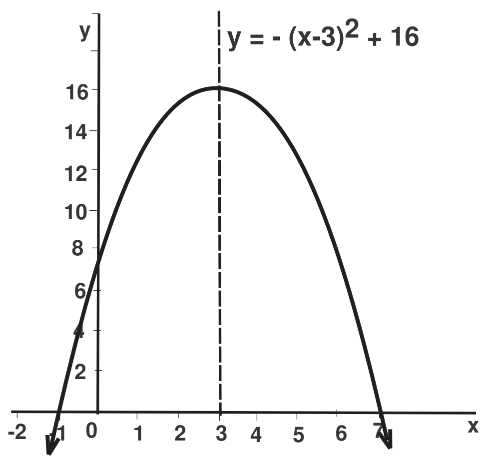

Figure 2.3

■

Problem 2.15. Find the vertex and axis of the parabola $y = x^2 - 6x + 9$ and then draw the parabola.

Solution. Given

$$y = x^2 - 6x + 9 = (x-3)^2$$

Comparing with the equation 2.1 we get $a = 1,\ h = 3,\ k = 0$. So axis is $x = h$ ie $x = 3$, vertex is $(h, k) = (3, 0)$, Here $a = 1 > 0$ so parabola is opens up. Thus we get graph of above quadratic function is a parabola with vertex at $(3, 0)$, symmetrical about the axis $x = 3$ and it opens up.

x	0	1	2	3	4	5	6
$f(x) = (x-3)^2$	9	4	1	0	1	4	9

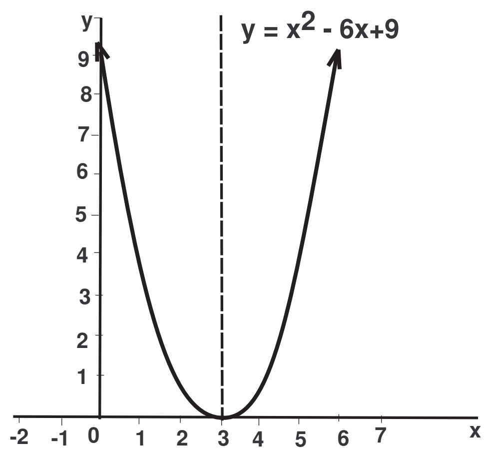

Figure 2.4

■

Problem 2.16. Find the vertex and axis of the parabola $y = x^2 - 8x + 19$ and then draw the parabola.

Solution. Given $y = x^2 - 8x + 19$, we first convert this into the form (2.1) using the procedure of completing the square. Here $b = -8$, $\frac{b}{2} = -4$ and $(-4)^2 = 16$. Adding and subtracting 16, we have

$$y = x^2 - 8x + 16 - 16 + 19$$

$$y = (x-4)^2 + 3$$

Comparing with 2.1 we get $a = 1, h = 4, k = 3$. So axis is $x = h$, that is $x = 4$, vertex is $(h, k) = (4, 3)$, Here $a = 1 > 0$ so parabola is opens up. Thus we get graph of above quadratic function is a parabola with vertex at $(4, 3)$, symmetrical about the axis $x = 4$ and it opens up.

x	2	3	4	5	6
$f(x) = x^2 - 8x + 9$	7	4	0	4	7

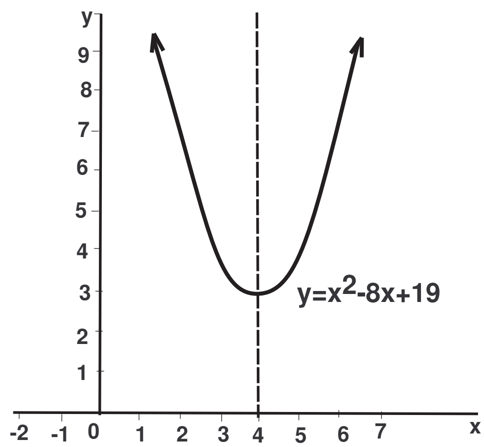

Figure 2.5

■

Drawing the graph of a rational function is made easier by finding the asymptotes. The *vertical asymptote* is the line $x = k$, where k is found by setting denominator equal to zero and solve for x. The *horizontal asymptote* is the line $y = m$, where m is found by first solving the original equation for x, set its denominator equal to zero and then solve for y. In Figure 2.2 as $x \to 0$, the graph approaches the y- axis. So y- axis is a vertical asymptote. As $x \to \infty$, the graph approaches the x- axis. So x- axis is a horizontal asymptote.

Problem 2.17. Draw a rough sketch of the graphs of the following rational functions by finding (1) the vertical asymptote,(2) the horizontal asymptote, and then (3) selecting a number of representative points on each graph to determine its shape.

(a) $y = \dfrac{5}{x-2}$

(b) $y = \dfrac{x+2}{x-5}$

Solution.

(a) (1) Setting denominator equal to zero and solving for x gives,

$$\begin{aligned} x-2 &= 0 \\ x &= 2 \end{aligned}$$

. So the vertical asymptote is $x=2$.

(2) Solving the original equation for x,

$$\begin{aligned} y &= \frac{5}{x-2} \\ y(x-2) &= 5 \\ yx-2y &= 5 \\ yx &= 5+2y \\ x &= \frac{5+2y}{y} \end{aligned}$$

. Setting the denominator equal to zero and solving for y,

$$y=0$$

. So the horizontal asymptote is $y=0$.

(3)

x	-1	0	1	1.5	2.5	3	4
$y=\frac{5}{x-2}$	-1.66	-2.5	-5	-10	10	5	2.5

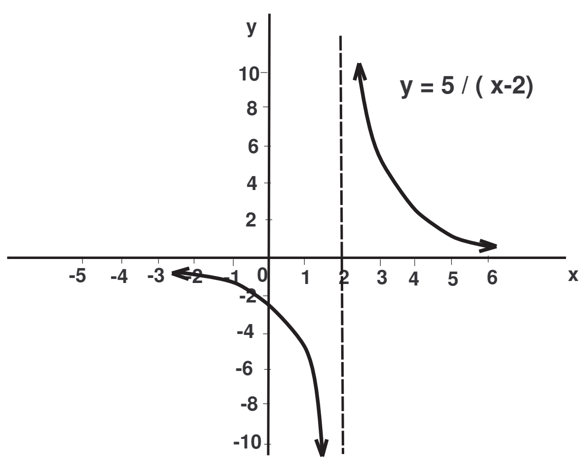

Figure 2.6

(b) (1) Vertical asymptote is $x = 5$

(2) Solving the original equation for x,

$$\begin{aligned} y &= \frac{x+2}{x-5} \\ y(x-5) &= x+2 \\ yx - 5y &= x+2 \\ yx - x &= 2+5y \\ x(y-1) &= 2+5y \\ x &= \frac{2+5y}{y-1} \end{aligned}$$

. Setting the denominator equal to zero and solving for y,

$$\begin{aligned} y - 1 &= 0 \\ y &= 1 \end{aligned}$$

. So the horizontal asymptote is $y = 1$.

(3)

x	1	3	4	6	7	9
$y = \frac{x+2}{x-5}$	-0.75	-2.5	-6	8	4.5	2.75

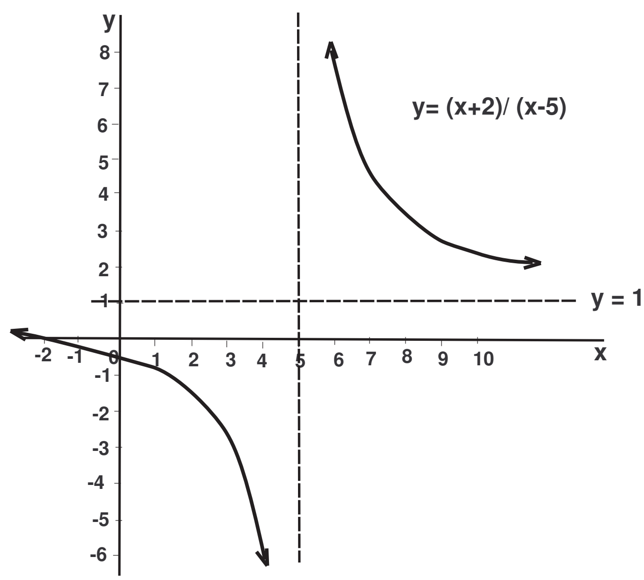

Figure 2.7

■

Problem 2.18. Given the following total revenue $R(x)$ and total cost $C(x)$ functions, express profit π as an explicit function of x and determine the maximum level of profit by finding the vertex of $\pi(x)$. Find the x-intercepts of the graph but do not graph.

(a) $R(x) = 600x - 5x^2,\ C(x) = 100x + 10,500$

(b) $R(x) = 280x - 2x^2,\ C(x) = 60x + 5,600$

Solution.

(a)

$$\begin{aligned} \pi(x) &= R(x) - C(x) \\ &= 600x - 5x^2 - (100x + 10,500) \\ &= 600x - 5x^2 - 100x - 10,500 \\ &= -5x^2 + 500x - 10,500 \\ &= -5(x^2 - 100x + 2,100) \end{aligned}$$

Completing the square,

$$\begin{aligned} \pi(x) &= -5(x^2 - 100x + 2,100) \\ &= -5(x^2 - 100x + (-50)^2 - (-50)^2 + 2,100) \\ &= -5((x-50)^2 - 2500 + 2100) \\ &= -5(x-50)^2 - 400) \\ &= -5(x-50)^2 + 2000 \end{aligned}$$

Comparing with (2.1) we get $a = -5,\ h = 50,\ k = 2000$. So vertex is $(h, k) = (50, 2000)$, Maximum profit $\pi(50) = 2000$.
To find x-intercepts we put $\pi(x) = 0$.

$$\begin{aligned} \pi(x) &= 0 \\ \Rightarrow -5(x^2 - 100x + 2,100) &= 0 \\ -5(x-30)(x-70) &= 0 \end{aligned}$$

$\Rightarrow x = 30\ \ x = 70$. So x-intercepts are $(30, 0),\ (70, 0)$.

(b)

$$\begin{aligned}
\pi(x) &= R(x) - C(x) \\
&= 280x - 2x^2 - (60x + 5,600) \\
&= 280x - 2x^2 - 60x - 5,600 \\
&= -2x^2 + 220x - 5,600 \\
&= -2(x^2 - 110x + 2,800)
\end{aligned}$$

Completing the square,

$$\begin{aligned}
\pi(x) &= -2(x^2 - 110x + 2,800) \\
&= -5(x^2 - 100x + (-55)^2 - (-55)^2 + 2,800) \\
&= -2((x - 55)^2 - 3025 + 2,800) \\
&= -2(x - 55)^2 - 225) \\
&= -2(x - 55)^2 + 450
\end{aligned}$$

Comparing with 2.1 we get $a = -2,\ h = 55,\ k = 450$. So vertex is $(h, k) = (55, 450)$, Maximum profit $\pi(55) = 450$.
To find x-intercepts we put $\pi(x) = 0$.

$$\begin{aligned}
\pi(x) &= 0 \\
\Rightarrow -2(x^2 - 110x + 2,800) &= 0 \\
-2(x - 40)(x - 70) &= 0
\end{aligned}$$

$\Rightarrow x = 40\ \ x = 70$. So x-intercepts are $(40, 0),\ (70, 0)$

■

2.6 Exercises

1. Given $f(x) = 2x^3 - 4x^2 + 7x - 10$, find $f(2)$ and $f(-3)$.

2. Given $f(x) = x^2 + 6x + 8$, find $f(a)$ and $f(a + 3)$.

3. Identify the domain of the following functions:
   (a) $\dfrac{x}{x^2 - 36}$ (b) $\dfrac{3x}{\sqrt{8 - x}}$

4. Identify the range of the functions $y = -5x$. $(1 \leq x \leq 4)$

5. If $f(x) = 6x - 5$ and $g(x) = 8x - 3$, then find $(f + g)(x), (f - g)(x), (f \cdot g)(x)$ and $(f \div g)(x)$

6. If $f(x) = x^2 + 6$ and $g(x) = 3x - 7$, then find $(f + g)(x), (f - g)(x), (f \cdot g)(x)$ and $(f \div g)(x)$

7. If $f(x) = x^3$, $g(x) = x^2 - 2x + 5$ and $h(x) = \dfrac{x}{x + 4}$, where $(x \neq -4)$ find: (a) $g[f(x)]$ (b) $f[h(x)]$ (c) $h[f(x)]$ (d) $h[g(x)]$

8. A widow receives \$5,600 a year from \$50,000 placed in two bonds, one paying 10 percent and the other 12 percent. How much does she have invested in each bond?

9. Graph the following quadratic functions and identify the vertex and axis of each:

   (a) $f(x) = 5 - x^2$

   (b) $f(x) = x^2 + 10x + 25$

   (c) $f(x) = x^2 + 12x + 41$

   (d) $f(x) = -x^2 - 4x - 1$

10. Draw a rough sketch of the graphs of the following rational functions by finding (1) the vertical asymptote, (2) the horizontal asymptote, and then (3) selecting a number of representative points on each graph to determine its shape.

    (a) $y = \dfrac{4}{x + 3}$

    (b) $y = \dfrac{-2}{x + 5}$

    (c) $y = \dfrac{3x + 2}{4x - 6}$

    (d) $y = \dfrac{6x + 1}{2x - 4}$

Chapter 3

The Derivative

3.1 Limits

$x \to a$ may be read as "x tends to a "or "x approaches a ". This means that x takes those values which are either less than a or greater than a but the numerical difference between x and a can be made as small as we please.

Let $f(x)$ be a function of x. Then $f(x)$ is said to approach L if the difference between $f(x)$ and L can be made as small as possible by taking x sufficiently near to a.That is, when x is very close to a, $f(x)$ also becomes close to L.
we write $\lim\limits_{x \to a} f(x) = L$.

Assuming that $\lim\limits_{x \to a} f(x)$ and $\lim\limits_{x \to a} g(x)$ both exist, the rules of limits are given below.

1. $\lim\limits_{x \to a} k = k$ (k = a constant)
2. $\lim\limits_{x \to a} x^n = a^n$ (n = a positive integer)
3. $\lim\limits_{x \to a} kf(x) = k \lim\limits_{x \to a} f(x)$ (k = a constant)
4. $\lim\limits_{x \to a}[f(x) \pm g(x)] = \lim\limits_{x \to a} f(x) \pm \lim\limits_{x \to a} g(x)$
5. $\lim\limits_{x \to a}[f(x) \cdot g(x)] = \lim\limits_{x \to a} f(x) \cdot \lim\limits_{x \to a} g(x)$
6. $\lim\limits_{x \to a}[f(x) \div g(x)] = \lim\limits_{x \to a} f(x) \div \lim\limits_{x \to a} g(x), \quad [\lim\limits_{x \to a} g(x) \neq 0]$
7. $\lim\limits_{x \to a}[f(x)]^n = [\lim\limits_{x \to a} f(x)]^n \quad (n > 0)$

Note 3.1. (a) For all polynomials $f(x)$, $\lim_{x\to a} f(x) = f(a)$

(b) For all rational functions $f(x) = g(x)/h(x)$ where $g(x)$ and $h(x)$ are polynomials and $h(x) \neq 0$, $\lim_{x\to a} f(x) = f(a)$

Problem 3.1. Use the rules of limits to find the limits for the following functions

(a) $\lim_{x\to 5}(3x^2 - 6x + 8)$

(b) $\lim_{x\to 7}[x^2(x-5)]$

(c) $\lim_{x\to 3} \dfrac{4x^2 - 9x}{x+7}$

(d) $\lim_{x\to 2} \sqrt{6x^2+1}$

Solution.

(a) $\lim_{x\to 5}(3x^2 - 6x + 8) = 3(5)^2 - 6(5) + 8 = 3(25) - 30 + 8 = 75 - 30 + 8 = 53$

(b)

$$\begin{aligned}
\lim_{x\to 7}[x^2(x-5)] &= \lim_{x\to 7} x^2 \cdot \lim_{x\to 7}(x-5) \\
&= (7)^2 \cdot (7-5) \\
&= 49 \cdot 2 \\
&= 98
\end{aligned}$$

(c)

$$\begin{aligned}
\lim_{x\to 3} \frac{4x^2 - 9x}{x+7} &= \frac{\lim_{x\to 3}(4x^2 - 9x)}{\lim_{x\to 3}(x+7)} \\
&= \frac{4(3)^2 - 9(3)}{3+7} \\
&= \frac{36 - 27}{10} \\
&= \frac{9}{10}
\end{aligned}$$

(d)

$$\begin{aligned}\lim_{x\to 2}\sqrt{6x^2+1} &= \lim_{x\to 2}(6x^2+1)^{1/2}\\ &= [\lim_{x\to 2}(6x^2+1)]^{1/2}\\ &= [6(2)^2+1]^{1/2}\\ &= (25)^{1/2}\\ &= 5\end{aligned}$$

■

Problem 3.2. Find the following limits:

(a) $\lim_{x\to 3}\dfrac{x-3}{x^2-9}$

(b) $\lim_{x\to 4}\dfrac{x+4}{x^2-16}$

(c) $\lim_{x\to 5}\dfrac{x^2-x-20}{x^2-25}$

(d) $\lim_{x\to 9}\dfrac{x^2+81}{x-9}$

Solution.

(a) Here the limit of the denominator is zero, so rule 6. can not be used. But we can find limit by factoring and cancelling the mutual terms.

$$\lim_{x\to 3}\frac{x-3}{x^2-9} = \lim_{x\to 3}\frac{x-3}{(x-3)(x+3)} = \lim_{x\to 3}\frac{1}{x+3} = \frac{1}{6}$$

(b) Here the limit of the denominator is zero, so rule 6. can not be used. But we can find limit by factoring and cancelling the mutual terms.

$$\lim_{x\to 4}\frac{x+4}{x^2-16} = \lim_{x\to 4}\frac{x+4}{(x-4)(x+4)} = \lim_{x\to 4}\frac{1}{x-4}$$

The limit does not exist.

(c) Since the limit of denominator equals zero, we factor

$$\lim_{x\to 5}\frac{x^2-x-20}{x^2-25} = \lim_{x\to 5}\frac{(x-5)(x+4)}{(x-5)(x+5)} = \lim_{x\to 5}\frac{x+4}{x+5} = \frac{9}{10}$$

(d) With the limit of the denominator equal to zero, we try to factor the numerator but see that it is impossible. We conclude, therefore that the limit does not exist.

■

Note 3.2. If x takes values which are close to a and always remains on the left of a, that is the value of x is less than a, then we say that x approaches a from left and we write $x \to a^-$. So $x \to a^-$ implies $x < a$.

If x takes values which are close to a and always remains on the right of a, that is the values of x is greater than a, then we say that x approaches a from right and we write $x \to a^+$. So $x \to a^+$ implies $x > a$.

Problem 3.3. Find the limits of the following functions:

(a) $\lim_{x \to 0} \frac{1}{x}$

(b) $\lim_{x \to \infty} \frac{1}{x}$

(c) $\lim_{x \to -\infty} \frac{1}{x}$

(d) $\lim_{x \to \infty} \frac{2x^2 - 3x}{5x^2 - 12}$

(e) $\lim_{x \to \infty} \frac{7x^3 - 5x^2 + 12x}{4x^2 + 9x}$

Solution.

(a) Note that as x approaches zero from right ($x \to 0^+$), $f(x)$ approaches positive infinity; as x approaches zero from left ($x \to 0^-$), $f(x)$ approaches negative infinity. If a limit approaches either positive or negative infinity, the limit does not exist and is written,

$$\lim_{x \to 0^+} \frac{1}{x} = \infty \qquad \lim_{x \to 0^-} \frac{1}{x} = -\infty$$

The limit does not exist.

(b) As x approaches ∞, $f(x)$ approaches zero, so the limit exist and we write

$$\lim_{x \to \infty} \frac{1}{x} = 0$$

(c) As x approaches $-\infty$, $f(x)$ approaches zero, so the limit exist and we write

$$\lim_{x\to-\infty}\frac{1}{x}=0$$

(d) As $x \to \infty$, both numerator and denominator become infinite. Divide all terms by the highest power of x which appears in the function. Here dividing all terms by x^2 leaves,

$$\lim_{x\to\infty}\frac{2x^2-3x}{5x^2-12}=\lim_{x\to\infty}\frac{2-(3/x)}{5-(12/x^2)}=\frac{2-(0)}{5-(0)}=\frac{2}{5}$$

(e) As $x \to \infty$, both numerator and denominator become infinite. Divide all terms by the highest power of x which appears in the function. Here dividing all terms by x^3 leaves

$$\lim_{x\to\infty}\frac{7x^3-5x^2+12x}{4x^2+9x}=\lim_{x\to\infty}\frac{7-(5/x)+(12/x^2)}{(4/x)+(9/x^2)}=\frac{7-0+0}{0+0}$$

Here denominator equal to zero. So the limit does not exist.

■

Problem 3.4. Find the following limits:

(a) $\lim_{x\to2}\frac{6x+1}{2x-4}$

(b) $\lim_{x\to\infty}\frac{6x+1}{2x-4}$

(c) $\lim_{x\to5}\frac{x+2}{x-5}$

(d) $\lim_{x\to\infty}\frac{5}{x-2}$

Solution.

(a) As $x \to 2$, the denominator approaches zero. Hence

$$\lim_{x\to2}\frac{6x+1}{2x-4}=\infty$$

(b)

$$\lim_{x\to\infty}\frac{6x+1}{2x-4}=\lim_{x\to\infty}\frac{6+(1/x)}{2-(4/x)}=\frac{3+0}{4-0}=\frac{3}{4}$$

(c) As $x\to 5$, the denominator approaches zero. Hence

$$\lim_{x\to 5}\frac{x+2}{x-5}=\infty$$

(d) As $x\to\infty$, the denominator approaches infinity, and

$$\lim_{x\to\infty}\frac{5}{x-2}=0$$

■

Problem 3.5. Find the limits of the following functions involving radicals:

(a) $\lim\limits_{x\to 0}\dfrac{7}{\sqrt{x+144}-2}$

(b) $\lim\limits_{x\to 36}\dfrac{\sqrt{x}-6}{x-36}$

(c) $\lim\limits_{x\to 3}\dfrac{\sqrt{x}-\sqrt{3}}{x-3}$

Solution.

(a) $\lim\limits_{x\to 0}\dfrac{7}{\sqrt{x+144}-2}=\dfrac{7}{12-2}=\dfrac{7}{10}$

(b) Here the limit of the denominator equal to zero. Here we multiply both numerator and denominator by $\sqrt{x}+6$,

$$\begin{aligned}\lim_{x\to 36}\frac{\sqrt{x}-6}{x-36} &= \lim_{x\to 36}\frac{\sqrt{x}-6}{x-36}\cdot\frac{\sqrt{x}+6}{\sqrt{x}+6}\\ &= \lim_{x\to 36}\frac{x+6\sqrt{x}-6\sqrt{x}-36}{(x-36)(\sqrt{x}+6)}\\ &= \lim_{x\to 36}\frac{(x-36)}{(x-36)(\sqrt{x}+6)}\\ &= \lim_{x\to 36}\frac{1}{\sqrt{x}+6}\\ &= \frac{1}{\sqrt{36}+6}=\frac{1}{6+6}=\frac{1}{12}\\ \therefore\quad \lim_{x\to 36}\frac{\sqrt{x}-6}{x-36} &= \frac{1}{12}\end{aligned}$$

(c) Here the limit of the denominator equal to zero. Here we multiply both numerator and denominator by $\sqrt{x}+\sqrt{3}$

$$\begin{aligned}
\lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} &= \lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3}\cdot\frac{\sqrt{x}+\sqrt{3}}{\sqrt{x}+\sqrt{3}}\\
&= \lim_{x\to 3}\frac{x-3}{(x-3)(\sqrt{x}+\sqrt{3})}\\
&= \lim_{x\to 3}\frac{1}{\sqrt{x}+\sqrt{3}}\\
&= \frac{1}{\sqrt{3}+\sqrt{3}}=\frac{1}{2\sqrt{3}}\\
\therefore\quad \lim_{x\to 3}\frac{\sqrt{x}-\sqrt{3}}{x-3} &= \frac{1}{2\sqrt{3}}
\end{aligned}$$

■

3.2 Continuity

The graph of a continuous function has no break; it can be drawn without lifting pencil from paper. A function f is *continuous* at $x=a$ if all three of the following conditions hold:

(1) $f(x)$ is defined, that is, exists, at $x=a$.

(2) $\lim_{x\to a} f(x)$ exists.

(3) $\lim_{x\to a} f(x)=f(a)$.

From the *properties of continuity*, it can be shown that for all real x:

1. All polynomial functions are continuous.
2. All rational functions are continuous except where undefined, that is where their denominators equal zero.
3. Suppose $f(x)$ and $g(x)$ are continuous at a point, at that point $f(x)\pm g(x)$ is continuous.
4. $f(x)\cdot g(x)$ is continuous.

5. $f(x) \div g(x)$ is continuous $[g(x) \neq 0]$.

6. $\sqrt[n]{f(x)}$ is continuous [whenever $\sqrt[n]{f(x)}$ is defined].

Problem 3.6. Check whether the following functions are continuous at the specified points:

(a) $f(x) = 14x + 6 \quad \text{at } x = 5$

(b) $f(x) = \dfrac{x^2 + 7x + 5}{x - 2} \quad \text{at } x = 3$

(c) $f(x) = \dfrac{x - 2}{x^2 - 4} \quad \text{at } x = 2$

Solution.

(a) (1) $f(5) = 14(5) + 6 = 70 + 6 = 76$ .The function is defined at $x = 5$.

(2) $\lim_{x\to 5} f(x) = \lim_{x\to 5}(14x + 6) = 14(5) + 6 = 76.$

(3) $\lim_{x\to 5} f(x) = 76 = f(5)$. So $f(x)$ is continuous.

(b) (1) $f(3) = \dfrac{(3)^2 + 7(3) + 5}{(3) - 2} = \dfrac{35}{1} = 35$. The function is defined at $x = 3$.

(2) $\lim_{x\to 3} f(x) = \lim_{x\to 3} \dfrac{x^2 + 7x + 5}{x - 2} = \dfrac{(3)^2 + 7(3) + 5}{(3) - 2} = \dfrac{35}{1} = 35.$

(3) $\lim_{x\to 3} f(x) = 35 = f(3)$. So $f(x)$ is continuous.

(c) (1)

$$f(2) = \frac{2 - 2}{(2)^2 - 4}$$

with the denominator equal to zero, $f(x)$ is not defined at $x = 2$. So $f(x)$ is not continuous at $x = 2$ even though limit exist at $x = 2$.

(2) $\lim_{x\to 2} f(x) = \lim_{x\to 2} \dfrac{x - 2}{x^2 - 4} = \lim_{x\to 2} \dfrac{x - 2}{(x - 2)(x + 2)} = \lim_{x\to 2} \dfrac{1}{x + 2} = \dfrac{1}{4}.$

(3) $\lim_{x\to 2} f(x) = \dfrac{1}{4} \neq f(2)$. So $f(x)$ is discontinuous at $x = 2$.

■

Problem 3.7. Prove that a polynomial function is continuous for any real value a of x, given $f(x) = k_0x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_{n-1}x + k_n$

Solution.

(1) $f(a) = k_0a^n + k_1a^{n-1} + k_2a^{n-2} + \cdots + k_{n-1}a + k_n$.

(2)

$$\begin{aligned}\lim_{x\to a} f(x) &= \lim_{x\to a}(k_0x^n + k_1x^{n-1} + k_2x^{n-2} + \cdots + k_{n-1}x + k_n) \\ &= \lim_{x\to a} k_0x^n + \lim_{x\to a} k_1x^{n-1} + \cdots + \lim_{x\to a} k_{n-1}x + \lim_{x\to a} k_n \\ &= k_0 \lim_{x\to a} x^n + k_1 \lim_{x\to a} x^{n-1} + \cdots + k_{n-1} \lim_{x\to a} x + \lim_{x\to a} k_n \\ &= k_0a^n + k_1a^{n-1} + k_2a^{n-2} + \cdots + k_{n-1}a + k_n.\end{aligned}$$

(3) $\lim_{x\to a} f(x) = f(a)$. So the polynomial is continuous at $x = a$.

■

Problem 3.8. For the following functions find the points of discontinuity:

(a) $f(x) = 7x^2 - 4x + 23$

(b) $g(x) = \dfrac{13x}{(x+2)(x-4)}$

(c) $h(x) = \dfrac{x+7}{x^2-49}$

(d) $f(x) = \sqrt{14-x}$

Solution.

(a) Since $f(x) = 7x^2 - 4x + 23$ is a polynomial it is continuous for all real values of x. So there is no point of discontinuity.

(b) Note that $g(x) = \dfrac{13x}{(x+2)(x-4)}$ is not defined for any value of x which would make the denominator equal to zero. Here denominator becomes zero when x takes values -2 and 4. So points of discontinuities are $x = -2$ and $x = 4$

(c) $h(x) = \dfrac{x+7}{x^2-49} = \dfrac{x+7}{(x+7)(x-7)} = \dfrac{1}{x-7}$.

Note that even though $h(x)$ can be reduced to $\dfrac{1}{x-7}$, if $x = -7$, the denominator of the original function is zero and $h(x)$ is undefined. Hence $h(x)$ is discontinuous at both 7 and -7.

(d) Since a square root is defined only for non negative numbers, $f(x)$ is not defined for $x > 14$. So points of discontinuities are $x > 14$.

■

3.3 The Slope of a Curvilinear Function

Linear functions are easy to use because the rate of change in the dependent variable as the independent variable changes is constant. For many relations, however, the rate of change in y as x changes is not constant. Functions for which the rate of change, or slope, varies are called *curvilinear functions*. As the name suggests, the graph of a curvilinear function is a curve rather than a straight line.

The slope of a curve varies continuously with movements along the curve. Geometrically, the slope of a curvilinear function at a given point is measured by the slope of a line drawn tangent (a tangent line to a circle is a straight line that touches the circle at only one point) to the function at that point.

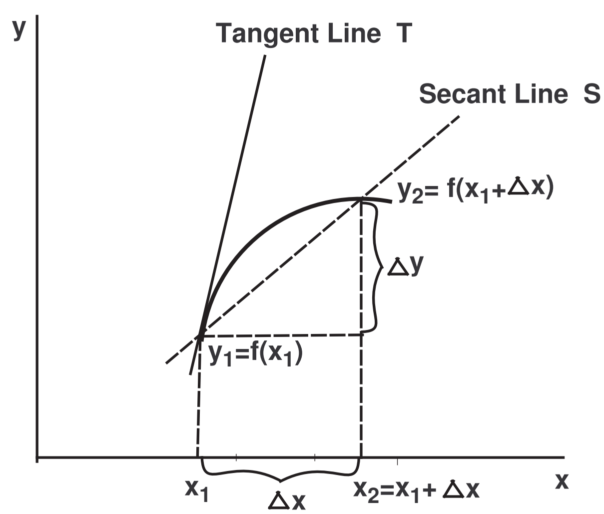

Figure 3.1

A *secant line S* is a straight line that intersects the graph of a function at two points. From above Figure

$$Slope\ S = \frac{y_2 - y_1}{x_2 - x_1}$$

By letting $x_2 = x_1 + \Delta x$ and $y_2 = f(x_1 + \Delta x)$, the slope of the secant line can also be expressed by

$$Slope\ S = \frac{f(x_1 + \Delta x) - f(x_1)}{(x_1 + \Delta x) - x_1} = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x}$$

If the distance between x_2 and x_1 is made smaller and smaller, that is if $\Delta x \to 0$, the secant line draws closer and closer to the tangent line. If the slope of the secant line approaches a limit as $\Delta x \to 0$, the limit is the slope of the tangent line T, hence the slope of the function itself at the point, and is written

$$Slope\ T = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{3.1}$$

In many texts h is used in place of Δx, giving,

$$Slope\ T = \lim_{h \to 0} \frac{f(x_1 + h) - f(x_1)}{h} \tag{3.2}$$

Problem 3.9. For the following functions find (1) The slope (2) The equation of the tangent line at the given point:

(a) $y = x^2 - 2$ at $(3, 7)$

(b) $y = 2x^2 - 3$ at $(2, 5)$

(c) $y = \dfrac{1}{x}$ at $(1, 1)$

Solution.

(a) (1) Using (3.1) for the formula for the slope of a tangent line to find the slope S of the curve at the given point:

$$Slope\ S = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Substituting $x^2 - 2$ for $f(x)$,

$$Slope\ S = \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 - 2] - (x^2 - 2)}{\Delta x}$$

Simplifying above equation we get,

$$\begin{aligned}
Slope\ S &= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + (\Delta x)^2 - 2) - (x^2 - 2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{(x^2 + 2x\Delta x + (\Delta x)^2 - 2 - x^2 + 2)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 2x + \Delta x \\
&= 2x
\end{aligned}$$

So Slope $S = 2x$.
(2) At $x = 3, S = 2(3) = 6$, which is the slope of the function at $(3, 7)$. Substituting $m = 6$ in point-slope formula $y - y_1 = m(x - x_1)$ we get,

$$\begin{aligned}
y - 7 &= 6(x - 3) \\
y - 7 &= 6x - 18 \\
y &= 6x - 18 + 7 \\
y &= 6x - 11
\end{aligned}$$

So equation of tangent line at $(3, 7)$ is $y = 6x - 11$.

(b) (1)

$$Slope\ S = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Substituting $2x^2 - 3$ for $f(x)$,

$$Slope\ S = \lim_{\Delta x \to 0} \frac{[2(x + \Delta x)^2 - 3] - (2x^2 - 3)}{\Delta x}$$

Simplifying above equation we get,

$$Slope\ S = \lim_{\Delta x \to 0} \frac{2(x^2 + 2x\Delta x + (\Delta x)^2) - 3 - (2x^2 - 3)}{\Delta x}$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{2x^2 + 4x\Delta x + 2(\Delta x)^2 - 3 - 2x^2 + 3)}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{4x\Delta x + 2(\Delta x)^2}{\Delta x} \\
&= \lim_{\Delta x \to 0} 4x + 2\Delta x \\
&= 4x
\end{aligned}
$$

So Slope $S = 4x$.
(2) At $x = 2$, $S = 4(2) = 8$, which is the slope of the function at $(2, 5)$. Substituting $m = 8$ in point-slope formula $y - y_1 = m(x - x_1)$ we get,

$$
\begin{aligned}
y - 5 &= 8(x - 2) \\
y - 5 &= 8x - 16 \\
y &= 8x - 16 + 5 \\
y &= 8x - 11
\end{aligned}
$$

So equation of tangent line at $(2, 5)$ is $y = 8x - 11$.

(c)

$$\text{Slope } S = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Substituting $\frac{1}{x}$ for $f(x)$,

$$\text{Slope } S = \lim_{\Delta x \to 0} \frac{\frac{1}{x + \Delta x} - \frac{1}{x}}{\Delta x}$$

Simplifying above equation we get,

$$
\begin{aligned}
\text{Slope } S &= \lim_{\Delta x \to 0} \frac{\frac{x - (x + \Delta x)}{(x + \Delta x)x}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{x - (x + \Delta x)}{(x + \Delta x)x\Delta x}
\end{aligned}
$$

$$
\begin{aligned}
&= \lim_{\Delta x \to 0} \frac{x - x - \Delta x}{(x + \Delta x)x\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-\Delta x}{(x + \Delta x)x\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{-1}{(x + \Delta x)x} \\
&= \lim_{\Delta x \to 0} \frac{-1}{x^2 + x\Delta x} \\
&= \frac{-1}{x^2}
\end{aligned}
$$

So Slope $S = \dfrac{-1}{x^2}$

(2) At $x = 1$, $S = \dfrac{-1}{(1)^2} = -1$, which is the slope of the function at $(1, 1)$. Substituting $m = -1$ in point-slope formula $y - y_1 = m(x - x_1)$ we get,

$$
\begin{aligned}
y - 1 &= -1(x - 1) \\
y - 1 &= -x + 1 \\
y &= -x + 1 + 1 \\
y &= -x + 2
\end{aligned}
$$

So equation of tangent line at $(1, 1)$ is, $y = -x + 2$.

■

3.4 Rates of Change

Given a function $y = f(x)$, the *average rate of change* from x_1 to x_2 is defined as the change in the dependent variable divided by the change in the independent variable:

$$\text{Average rate of change} = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

$$Average\ rate = \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{3.3}$$

As $\Delta x \to 0$, assuming the limit exists, the average rate of change approaches the *instantaneous rate of change*, which is defined by

$$Instantaneous\ rate = \lim_{\Delta x \to 0} \frac{f(x_1 + \Delta x) - f(x_1)}{\Delta x} \tag{3.4}$$

Problem 3.10. The laws of physics indicate that a lead pellet dropped from a bridge will fall a distance of y feet in t seconds, given by the formula $y = f(t) = 16t^2$, with velocity = distance divided by time, Estimate (a) The average velocity (AV) of the pellet between $t = 3$ and $t = 4$, (b) The average velocity for a small change in time Δt starting with $t = 3$ and (c) The instantaneous velocity at $t = 3$.

Solution.

(a) Average velocity $= \dfrac{\Delta y}{\Delta x} = \dfrac{y_2 - y_1}{x_2 - x_1} = \dfrac{f(4) - f(3)}{4 - 3}$

$$= \frac{16(4)^2 - 16(3)^2}{1} = \frac{256 - 144}{1} = 112$$

So average velocity of the pellet between $t = 3$ and $t = 4$ is 112 ft/sec.

(b) Average velocity for a small change in time, Δt, using (3.3) and starting from $t = 3$

$$\begin{aligned} Average\ velocity &= \frac{f(3 + \Delta t) - f(3)}{(3 + \Delta t) - 3} \\ &= \frac{16(3 + \Delta t)^2 - 16(3)^2}{\Delta t} \\ &= \frac{16[9 + 6\Delta t + (\Delta t)^2] - 16(9)}{\Delta t} \\ &= \frac{144 + 96\Delta t + 16(\Delta t)^2 - 144}{\Delta t} \\ &= \frac{96\Delta t + 16(\Delta t)^2}{\Delta t} \\ &= \frac{\Delta t(96 + 16(\Delta t))}{\Delta t} \\ &= 96 + 16\Delta t \quad ft/sec \end{aligned}$$

If at $t = 3,\ \Delta t = 1$, AV$= 96 + 16(1) = 112$ ft/sec ;

if $\Delta t = \frac{1}{4}$, AV $= 96 + 16(\frac{1}{4}) = 100$ ft/sec.

(c) If $\Delta t \to 0$, assuming the limit exist, we have from (3.4) the instantaneous velocity,

$$\begin{aligned} Instantaneous\ velocity\ at\ t = 3 &= \lim_{\Delta t \to 0} \frac{16(3+\Delta t)^2 - 16(3)^2}{\Delta t} \\ &= \lim_{\Delta t \to 0}(96 + 16\Delta t) \\ &= 96 \ \ ft/sec \end{aligned}$$

■

Problem 3.11. A trunk travels a distance of w miles in t hours given by the function $w(t) = 10t^2 + 5t$. Estimate (a) The average velocity (AV) of the pellet between $t = 2$ and $t = 3$, (b) The average velocity for a small change in time Δt starting with $t = 2$ and (c) The instantaneous velocity at $t = 2$.

Solution.

(a)

$$\begin{aligned} AV &= \frac{w(3) - w(2)}{3 - 2} \\ &= \frac{[10(3)^2 + 5(3)] - [10(2)^2 + 5(2)]}{3 - 2} \\ &= \frac{[10(9) + 15] - [10(4) + 10]}{1} \\ &= \frac{(90 + 15) - (40 + 10)}{1} \\ &= \frac{(105 - 50}{1} = 55 \ \ mph \end{aligned}$$

(b) Starting at $t = 2$,

$$AV = \frac{w(2 + \Delta t) - w(2)}{(2 + \Delta t) - 2}$$

$$\begin{aligned}
&= \frac{[10(2+\Delta t)^2 + 5(2+\Delta t)] - [10(2)^2 + 5(2)]}{\Delta t} \\
&= \frac{[10(4+4\Delta t+(\Delta t)^2) + 10 + 5\Delta t] - [10(4) + 10]}{\Delta t} \\
&= \frac{[40+40\Delta t+10(\Delta t)^2 + 10 + 5\Delta t] - [40 + 10]}{\Delta t} \\
&= \frac{[50+45\Delta t+10(\Delta t)^2] - [50]}{\Delta t} \\
&= \frac{50+45\Delta t+10(\Delta t)^2 - 50}{\Delta t} \\
&= \frac{45\Delta t+10(\Delta t)^2}{\Delta t} \\
&= 45 + 10\Delta t
\end{aligned}$$

If at $t = 2$, $\Delta t = 1$, AV$= 45 + 10(1) = 55$ mph; if $\Delta t = 0.5$, AV $= 45 + 10(0.5) = 45 + 5 = 50$ mph.

(c) If $\Delta t \to 0$, assuming the limit exist, we have from (3.4) the instantaneous velocity(IV)

$$\begin{aligned}
IV \;\; at \;\; t = 2 &= \lim_{\Delta t \to 0} \frac{[10(2+\Delta t)^2 + 5(2+\Delta t)] - [10(2)^2 + 5(2)]}{\Delta t} \\
&= \lim_{\Delta t \to 0} (45 + 10\Delta t) \\
&= 45 \;\; mph \;\; at \;\; t = 2
\end{aligned}$$

■

3.5 The Derivative

Given a function $y = f(x)$, the *derivative* of the function f at x, written $f'(x)$ or $\dfrac{dy}{dx}$ is defined as

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x} \tag{3.5}$$

provided the limit exists.

We read $f'(x)$ as "the derivative of f with repect to x ", or "f prime of x".

The derivative of a function, $f'(x)$ of simply f', is itself a function which measures the slope and the instantaneous rate of change of the original function $f(x)$ at a given point. The process of finding a derivative is called *differentiation*.

Problem 3.12. For each of the following functions use delta process to (1) find the derivative and then (2) Evaluate the derivative for the given value of x :

(a) $f(x) = 5x + 7$ at $x = 8$

(b) $f(x) = x^2 - 6$ at $x = -5$

(c) $f(x) = \dfrac{1}{3x + 7}$ at $x = -2$

(d) $f(x) = \sqrt{x}$ at $x = 16$

Solution.

(a) (1)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Substituting $5x + 7$ for $f(x)$,

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{[5(x + \Delta x) + 7] - (5x + 7)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[5x + 5\Delta x + 7 - (5x + 7)}{\Delta x} \end{aligned}$$

$$= \lim_{\Delta x \to 0} \frac{(5x + 5\Delta x + 7 - 5x - 7)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{5\Delta x}{\Delta x}$$

$$= \lim_{\Delta x \to 0} 5 = 5$$

So $f'(x) = 5$.

(2) Derivative at $x = 8$ is $f'(8) = 5$.

(b) (1)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Substituting $x^2 - 6$ for $f(x)$,

$$\begin{aligned} f'(x) &= \lim_{\Delta x \to 0} \frac{[(x + \Delta x)^2 - 6] - (x^2 - 6)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{[x^2 + 2x\Delta x + (\Delta x)^2 - 6] - (x^2 - 6)}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{x^2 + 2x\Delta x + (\Delta x)^2 - 6 - x^2 + 6}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{2x\Delta x + (\Delta x)^2}{\Delta x} \\ &= \lim_{\Delta x \to 0} 2x + \Delta x \\ &= 2x \end{aligned}$$

So $f'(x) = 2x$.

(2) Derivative at $x = -5$ is $f'(-5) = -10$.

(c) (1)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}$$

Substituting $\dfrac{1}{3x+7}$ for $f(x)$,

$$
\begin{aligned}
f'(x) &= \lim_{\Delta x \to 0} \frac{\dfrac{1}{3(x+\Delta x)+7} - \dfrac{1}{3x+7}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{\dfrac{(3x+7)-(3x+3\Delta x+7)}{(3x+3\Delta x+7)(3x+7)}}{\Delta x} \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{(3x+7)-(3x+3\Delta x+7)}{(3x+3\Delta x+7)(3x+7)}\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{3x+7-3x-3\Delta x-7}{(9x^2+21x+9x\Delta x+21\Delta x+21x+49)}\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{-3\Delta x}{(9x^2+42x+49)+9x\Delta x+21\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{1}{\Delta x}\left[\frac{-3\Delta x}{(3x+7)^2+9x\Delta x+21\Delta x}\right] \\
&= \lim_{\Delta x \to 0} \frac{-3}{(3x+7)^2+9x\Delta x+21\Delta x} \\
&= \frac{-3}{(3x+7)^2}
\end{aligned}
$$

So $f'(x) = \dfrac{-3}{(3x+7)^2}$.

(2) Derivative at $x = -2$ is $f'(-2) = \dfrac{-3}{[(3(-2)+7)]^2} = \dfrac{-3}{(-6+7)^2} = -3.$

(d) (1)

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x}$$

Denominator approaches zero as $\Delta x \to 0$, So we rationalize the numerator

$$f'(x) = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x}-\sqrt{x}}{\Delta x} \cdot \frac{\sqrt{x+\Delta x}+\sqrt{x}}{\sqrt{x+\Delta x}+\sqrt{x}}$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})}$$

$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}}$$

$$= \frac{1}{2\sqrt{x}}$$

So $f'(x) = \dfrac{1}{2\sqrt{x}}$.

(2) Derivative at $x = 16$ is $f'(16) = \dfrac{1}{2\sqrt{16}} = \dfrac{1}{2(4)} = \dfrac{1}{8}$.

■

3.6 Differentiability and Continuity

A function is *differentiable* at a point if the derivative exists at that point. For a function to be differentiable at a point, the function must (1) be continuous at that point and (2) have a unique tangent at that point. In Figure 3.2, the function is not differentiable at a and b because gaps exist in the function at those points and the derivative cannot be taken at any point where the function is discontinuous. Continuity need not imply differentiability. In Figure 3.2, the function is continuous at c, but it is not differentiable at c because at a sharp point or *cusp* an infinite number of tangent lines (and no one unique tangent line) can be drawn.

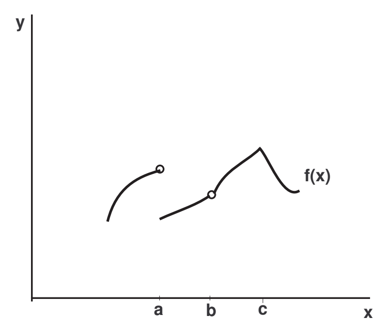

Figure 3.2

3.7 Applications for the Social Sciences, Business, and Economics

Economists are particularly interested in *marginal* rates of change, such as marginal cost, marginal revenue, marginal utility, marginal product, etc., all of which are measured mathematically by derivative.

Marginal cost is defined as the change in total cost associated with a small change in the quantity produced. It is measured by the derivative of the total cost function. If $C(x)$ is the total cost function, then $C'(x)$ is the marginal cost function. Similarly, if $R(x)$ is the total revenue function, then $R'(x)$ is the *marginal revenue* function.

Problem 3.13. Given the total cost function C (in dollars) of producing x pounds of fertilizer, $C(x) = 0.5x^2 + 1.5x + 8$, Find

(a) The average cost (AC) of production between $x = 4$ and $x = 6$.

(b) The AC of production for a small increase starting at $x = 4$.

(c) The marginal cost (MC) at $x = 4$.

Solution.

(a)

$$\begin{aligned}
AC &= \frac{C(6) - C(4)}{6 - 4} \\
&= \frac{[0.5(6)^2 + 1.5(6) + 8] - [0.5(4)^2 + 1.5(4) + 8]}{2} \\
&= \frac{[0.5(36) + 9 + 8] - [0.5(16) + 6 + 8]}{2} \\
&= \frac{(18 + 9 + 8) - (8 + 6 + 8)}{2} \\
&= \frac{35 - 22}{2} \\
&= \frac{13}{2} = 6.5
\end{aligned}$$

(b)

$$\begin{aligned}
AC &= \frac{C(x+\Delta x)-C(x)}{\Delta x} \\
&= \frac{[0.5(x+\Delta x)^2+1.5(x+\Delta x)+8]-[0.5x^2+1.5x+8]}{\Delta x} \\
&= \frac{[0.5(x^2+2x\Delta x+(\Delta x)^2)+1.5x+1.5\Delta x+8]-[0.5x^2+1.5x+8]}{\Delta x} \\
&= \frac{[0.5x^2+x\Delta x+0.5(\Delta x)^2+1.5x+1.5\Delta x+8]-[0.5x^2+1.5x+8]}{\Delta x} \\
&= \frac{0.5x^2+x\Delta x+0.5(\Delta x)^2+1.5x+1.5\Delta x+8-0.5x^2-1.5x-8}{\Delta x} \\
&= \frac{x\Delta x+0.5(\Delta x)^2+1.5\Delta x}{\Delta x} \\
&= x+0.5\Delta x+1.5 \\
&= x+1.5+0.5\Delta x
\end{aligned}$$

If at $x=4,\ \Delta x=1$, AC $=4+1.5+0.5(1)=6;\ \ \Delta x=2$, AC $=4+1.5+0.5(2)=6.5$.

(c) Marginal cost MC$=\lim\limits_{\Delta x\to 0}\dfrac{C(x+\Delta x)-C(x)}{\Delta x}$

$$\begin{aligned}
MC &= \lim_{\Delta x\to 0}\frac{[0.5(x+\Delta x)^2+1.5(x+\Delta x)+8]-[0.5x^2+1.5x+8]}{\Delta x} \\
&= \lim_{\Delta x\to 0}(x+1.5+0.5\Delta x) \\
&= x+1.5
\end{aligned}$$

At $x=4,\ MC=(4)+1.5=5.5$.

■

Problem 3.14. Given the total revenue function R from the sale of x units, $R(x) = 150x - 3x^2$, Find

(a) The average revenue (AR) of sales between $x = 15$ and $x = 20$.

(b) The AR of sales for a small increase of sales starting at $x = 15$.

(c) The marginal cost (MR) at $x = 15$.

Solution.

(a)

$$\begin{aligned} AR &= \frac{R(20) - R(15)}{20 - 15} \\ &= \frac{[150(20) - 3(20)^2] - [150(15) - 3(15)^2]}{5} \\ &= \frac{[3000 - 3(400)] - [2250 - 3(225)]}{5} \\ &= \frac{(3000 - 1200) - (2250 - 675)}{5} \\ &= \frac{1800 - 1575}{5} \\ &= \frac{225}{5} = 45 \end{aligned}$$

(b)

$$\begin{aligned} AR &= \frac{R(x + \Delta x) - R(x)}{\Delta x} \\ &= \frac{[150(x + \Delta x) - 3(x + \Delta x)^2] - [150x - 3x^2]}{\Delta x} \\ &= \frac{[150x + 150\Delta x - 3(x^2 + 2x\Delta x + (\Delta x)^2)] - [150x - 3x^2]}{\Delta x} \\ &= \frac{[150x + 150\Delta x - 3x^2 - 6x\Delta x - 3(\Delta x)^2] - [150x - 3x^2]}{\Delta x} \end{aligned}$$

$$= \frac{[150x + 150\Delta x - 3x^2 - 6x\Delta x - 3(\Delta x)^2 - 150x + 3x^2]}{\Delta x}$$

$$= \frac{150\Delta x - 6x\Delta x - 3(\Delta x)^2}{\Delta x}$$

$$= \frac{\Delta x(150 - 6x - 3\Delta x)}{\Delta x}$$

$$= 150 - 6x - 3\Delta x$$

If at $x = 15,\ \Delta x = 5$, AR $= 150 - 6(15) - 3(5) = 150 - 90 - 15 = 45$.

(c) Marginal revenue MR $= \dfrac{dR}{dx} = \lim\limits_{\Delta x \to 0} \dfrac{R(x + \Delta x) - R(x)}{\Delta x}$

$$MR = \frac{[150(x + \Delta x) - 3(x + \Delta x)^2] - [150x - 3x^2]}{\Delta x}$$

$$= \lim_{\Delta x \to 0}(150 - 6x - 3\Delta x)$$

$$= 150 - 6x$$

At $x = 15,\ MR = 150 - 6(15) = 150 - 90 = 60$.

■

3.8 Exercises

1. Use the rules of limits to find the limits for the following functions:

   (a) $\lim\limits_{x \to -4}(2x^2 + 5x - 22)$

   (b) $\lim\limits_{x \to 3}(5x^2 - 13)^{1/3}$

   (c) $\lim\limits_{x \to -4} \dfrac{5x^2 - 9x + 22}{2x^2 + 7}$

2. Find the following limits:

   (a) $\lim\limits_{x \to -3} \dfrac{x - 3}{x^2 - 9}$

(b) $\lim_{x \to -4} \frac{4+x}{16-x^2}$

(c) $\lim_{x \to 3} \frac{x^2-x-6}{x-3}$

3. Find the following limits:

(a) $\lim_{x \to \infty} \frac{x-11}{3x^2+5x+9}$

(b) $\lim_{x \to \infty} \frac{8x^3-5x^2+13x}{2x^3+7x^2-18x}$

4. Find the limits of the following functions involving radicals:

(a) $\lim_{x \to 0} \frac{4}{23-\sqrt{x+49}}$

(b) $\lim_{x \to 16} \frac{\sqrt{x}-4}{x-16}$

(c) $\lim_{x \to 9} \frac{\sqrt{x}-\sqrt{9}}{x-9}$

(d) $\lim_{x \to 0} \frac{\sqrt{x+25}-5}{x}$

5. For the following functions find the points of discontinuities:

(a) $g(x) = \frac{6x}{x-5}$

(b) $f(x) = \frac{x+6}{x^2+2x-24}$

(c) $h(x) = \sqrt{x-22}$

6. Find the slope of the function $y = x^2 + 10x + 25$, also find the equation of the tangent line at the point $(-3, 4)$

7. A particle moved by an accelerator a distance w miles in t seconds given by the function $w(t) = 25t^2 + 45t$. Find (a) the average velocity between $t = 1$ and $t = 3$, (b) the average velocity for a small change in time starting at $t = 1$, and (c) the instantaneous velocity at $t = 1$.

8. Given the supply curve $S(p) = 50x^2$, find (a) the average change in supply AS from $p = \$3$ to $p = \$5$, (b) the average change in supply for a small change in price at $p = 3$, and (c) the instantaneous change in supply IS at $p = 3$.

Chapter 4

Differentiation

4.1 Derivative Notation

The derivative of a function can be expressed in several different ways. If $y = f(x)$, the derivative can be expressed as

$$f'(x) \qquad y' \qquad \frac{dy}{dx} \qquad \frac{df}{dx} \qquad \frac{d}{dx}[f(x)] \qquad or \;\; D_x[f(x)]$$

If $y = \phi(t)$, the derivative can be written as

$$\phi'(t) \qquad y' \qquad \frac{dy}{dt} \qquad \frac{d\phi}{dt} \qquad \frac{d}{dt}[\phi(t)] \qquad or \;\; D_t[\phi(t)]$$

If the derivative of $y = f(x)$ is evaluated at $x = a$, proper notation includes $f'(a)$ and $\left.\frac{dy}{dx}\right|_a$.

Example 4.1. If $y = 2x^3 + 4x + 6$, the derivative can be written as

$$y' \qquad \frac{dy}{dx} \qquad \frac{d}{dx}[2x^2 + 4x + 6] \qquad or \;\; D_x[2x^2 + 4x + 6]$$

4.2 Rules of Differentiation

The process of finding the derivative of a function is known as *differentiation*

The Constant Function Rule

The derivative of a constant function $f(x) = k$, where k is a constant, is zero.

Example 4.2. If $f(x) = 5$ then $f'(x) = 0$; If $f(x) = 8$ then $f'(x) = 0$

The Linear Function Rule

The derivative of a linear function $f(x) = mx + b$, is equal to m, the coefficient of x.

Example 4.3. If $f(x) = 2x + 3$ then $f'(x) = 2$
If $f(x) = -5x + 12$ then $f'(x) = -5$
If $f(x) = \frac{1}{4}x + 7$ then $f'(x) = \frac{1}{4}$

The Power Function Rule

The derivative of a power function $f(x) = x^n$, where n is any real number, is equal to nx^{n-1}

Example 4.4.

$$\begin{aligned} If\ \ f(x) &= x^3, \qquad f'(x) = 3x^{3-1} = 3x^2 \\ If\ \ f(x) &= x^5, \qquad f'(x) = 5x^{5-1} = 5x^4 \\ If\ \ f(x) &= x^{\frac{3}{2}}, \qquad f'(x) = \frac{3}{2}x^{\frac{3}{2}-1} = \frac{3}{2}x^{\frac{1}{2}} \end{aligned}$$

The Rule for Constant Times a Function

If $f(x) = kg(x)$, where k is a real number and $g(x)$ is a differentiable function then $f'(x) = kg'(x)$

Example 4.5.

$$\begin{aligned} If\ \ f(x) &= 4x^3, \qquad f'(x) = 4(3x^{3-1}) = 12x^2 \\ If\ \ f(x) &= -7x^5, \qquad f'(x) = -7(5x^{5-1}) = -35x^4 \end{aligned}$$

The Rule for Sum and Differences

Let $f(x) = g(x) + h(x)$ where $g(x)$ and $h(x)$ are both differentiable functions. Then $f'(x) = g'(x) + h'(x)$, Similarly if $f(x) = g(x) - h(x)$, then $f'(x) = g'(x) - h'(x)$.

Example 4.6.

$$\begin{aligned} If\ \ f(x) &= 8x^4 + 4x^3, \quad f'(x) = 8(4x^{4-1}) + 4(3x^{3-1}) = 32x^3 + 12x^2 \\ If\ \ f(x) &= 3x^6 - 7x^5 - 2, \quad f'(x) = 3(6x^{6-1}) - 7(5x^{5-1}) - 0 = 18x^5 - 35x^4 \end{aligned}$$

Problem 4.1. Differentiate each of the following functions. Use the different notations:

(a) $f(x) = 13$ (b) $y = 7x - 2$ (c) $y = 7x^4$ (d) $y = 4x^{-3}$

(e) $f(x) = 25\sqrt{x}$

Solution.

(a) $f(x) = 13 \Rightarrow f'(x) = 0$ (Constant Rule)

(b) $y = 7x - 2 \Rightarrow \dfrac{dy}{dx} = 7$ (Linear Function Rule)

(c) $y = 7x^4 \Rightarrow y' = 7(4x^{4-1}) = 7(4x^3) = -28x^3$ (Power Function Rule)

(d) $y = 4x^{-3} \Rightarrow y' = 4(-3x^{-3-1}) = -12x^{-4} = \dfrac{-12}{x^4}$ (Power Function Rule)

(e) $f(x) = 25\sqrt{x}, \Rightarrow D_x(25\sqrt{x}) = D_x(25x^{\frac{1}{2}})$

. $= 25(\frac{1}{2}x^{[(1/2)-1]}) = 12.5x^{-1/2} = \dfrac{12.5}{\sqrt{x}}$ (Power Function Rule)

■

Problem 4.2. Use the rule for sums and differences to differentiate the following functions, treating the variable on the left as dependent variable and right as independent variable.

(a) $R = 6t^2 + 11t - 9$

(b) $p = 7q^5 - 9q^3$

(c) $u = \dfrac{1}{x^2} - \dfrac{6}{x^3} = x^{-2} - 6x^{-3}$

Solution.

(a) $\dfrac{dR}{dt} = 12t + 11$

(b) $\dfrac{dp}{dq} = 35q^4 - 27q^2$

(c) $u' = -2x^{-3} - 6(-3x^{-4}) = -2x^{-3} + 18x^{-4} = \dfrac{-2}{x^3} + \dfrac{18}{x^4}$

■

The Product Rule

If $f(x) = g(x) \cdot h(x)$, where $g(x)$ and $h(x)$ are both differentiable function then,

$$f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x) \tag{4.1}$$

Problem 4.3. Differentiate following functions using product rule:

(a) $y = 8x^3(5x - 2)$

(b) $y = x^2(x - 1)$

(c) $y = (6x^2 - 7)(3x^4)$

(d) $f(x) = (5x^3 + 8x^2)(4x^5 - 2)$

Solution.

(a) We have if $f(x) = g(x) \cdot h(x)$ then $f'(x) = g(x) \cdot h'(x) + h(x) \cdot g'(x)$
Let $g(x) = 8x^3$ and $h(x) = 5x - 2$. Then $g'(x) = 24x^2$ and $h'(x) = 5$. Substituting these values in the product rule formula,

$$y' = 8x^3(5) + (5x - 2)24x^2 = 40x^3 + 120x^3 - 48x^2 = 160x^3 - 48x^2$$

(b) $y' = x^2(1) + (x - 1)(2x) = x^2 + 2x^2 - 2x = 3x^2 - 2x$

(c)

$$\begin{aligned} y' &= (6x^2 - 7)(12x^3) + 3x^4(12x) \\ &= 72x^5 - 84x^3 + 36x^5 \\ &= 108x^5 - 84x^3 \end{aligned}$$

(d)

$$\begin{aligned} f'(x) &= (5x^3 + 8x^2)(20x^4) + (4x^5 - 2)(15x^2 + 16x) \\ &= (100x^7 + 160x^6) + (60x^7 + 64x^6 - 30x^2 - 32x) \\ &= 160x^7 + 224x^6 - 30x^2 - 32x \end{aligned}$$

■

The Quotient Rule

If $f(x) = \dfrac{g(x)}{h(x)}$, where $g(x)$ and $h(x)$ are both differentiable function and $h(x) \neq 0$ then,

$$f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2} \tag{4.2}$$

Problem 4.4. Differentiate following functions using quotient rule:

(a) $f(x) = \dfrac{2x^4}{5x - 6}$

(b) $y = \dfrac{7x^3}{4x + 9}$

(c) $y = \dfrac{4x^3 - 11}{3x^2 + 7}$

Solution.

(a) We have If $f(x) = \dfrac{g(x)}{h(x)}$ $\quad (h(x) \neq 0)$ then

$$f'(x) = \frac{h(x) \cdot g'(x) - g(x) \cdot h'(x)}{[h(x)]^2}$$

Let $g(x) = 2x^4$ and $h(x) = 5x - 6$; then $g'(x) = 8x^3$ and $h'(x) = 5$. Substituting these values in the quotient rule formula,

$$\begin{aligned} f'(x) &= \frac{(5x-6)(8x^3) - 2x^4(5)}{(5x-6)^2} \\ &= \frac{(40x^4 - 48x^3) - 10x^4}{(5x-6)^2} \\ &= \frac{40x^4 - 48x^3 - 10x^4}{(5x-6)^2} \\ &= \frac{30x^4 - 48x^3}{(5x-6)^2} \end{aligned}$$

(b) Let $g(x) = 7x^3$ and $h(x) = 4x + 9$; then $g'(x) = 21x^2$ and $h'(x) = 4$. Substituting these values in the quotient rule formula,

$$\begin{aligned} y' &= \frac{(4x+9)(21x^2) - 7x^3(4)}{(4x+9)^2} \\ &= \frac{(84x^3 + 189x^2) - 28x^3}{(4x+9)^2} \\ &= \frac{84x^3 + 189x^2 - 28x^3}{(4x+9)^2} \\ &= \frac{56x^3 + 189x^2}{(4x+9)^2} \end{aligned}$$

(c)

$$\begin{aligned} \frac{dy}{dx} &= \frac{(3x^2+7)(12x^2) - (4x^3 - 11)(6x)}{(3x^2+7)^2} \\ &= \frac{(36x^4 + 84x^2) - (24x^4 - 66x)}{(3x^2+7)^2} \\ &= \frac{36x^4 + 84x^2 - 24x^4 + 66x}{(3x^2+7)^2} \\ &= \frac{12x^4 + 84x^2 + 66x}{(3x^2+7)^2} \end{aligned}$$

■

The Generalized Power Function Rule

If $f(x) = [g(x)]^n$, where $g(x)$ is a differentiable function and n is any real number then,

$$f'(x) = n[g(x)]^{n-1} \cdot g'(x) \tag{4.3}$$

Problem 4.5. Given $y = (4x + 9)^2$, (a) find the derivative directly, using the generalized power function rule. (b) Simplify the function first by squaring it and then take the derivative. (c) Compare the results.

Solution.

(a) Let $g(x) = 4x + 9, g'(x) = 4$, and $n = 2$. Substitute these values in the generalized power function rule,

$$y' = 2(4x + 9)^{2-1} \cdot 4 = 8(4x + 9) = 32x + 72$$

(b) Square the function first and then take the derivative,

$$\begin{aligned} y &= (4x+9)^2 \\ y &= 16x^2+72x+81 \\ y' &= 32x+72 \end{aligned}$$

(c) The derivatives are identical but the generalized power function rule is faster and more practical for higher, negative, and fractional values of n.

■

Problem 4.6. Differentiate following functions using the generalized power function rule:

(a) $f(x) = (2x^3+7)^5$

(b) $y = \sqrt{16-x^2}$

Solution.

(a) Here $g(x) = 2x^3+7, g'(x) = 6x^2$ and $n = 5$. Substitute these values in the generalized power function rule,

$$y' = 5(2x^3+7)^{5-1} \cdot 6x^2 = 5(2x^3+7)^4 \cdot 6x^2 = 30x^2(2x^3+7)^4$$

(b) Convert the radical to a power function, then differentiate,

$$\begin{aligned} y &= (16-x^2)^{1/2} \\ y' &= \frac{1}{2}(16-x^2)^{-1/2} \cdot (-2x) \\ &= -x(16-x^2)^{-1/2} \\ &= \frac{-x}{\sqrt{16-x^2}} \end{aligned}$$

■

The Chain Rule

If $f(x) = g[h(x)]$, where $g(x)$ and $h(x)$ are both differentiable functions then

$$f'(x) = g'[h(x)] \cdot h'(x) \tag{4.4}$$

The chain rule is also called the *composite function rule* or *the function of a function rule*

Problem 4.7. Find the derivative of the following function using chain rule

(a) $f(x) = \sqrt{8x+9}$

(b) $f(x) = (12x+9)^4$

(c) $f(x) = 2(4x^2+9)^3$

Solution.

(a) Let $g(x) = \sqrt{x}$ and $h(x) = 8x+9$, then $g'(x) = \frac{1}{2}x^{-1/2}$, $h'(x) = 8$.
Note that $f(x) = \sqrt{8x+9} = \sqrt{h(x)} = g[h(x)]$
Substituting $h(x)$ for x in $g'(x)$,

$$g'[h(x)] = g'[(8x+9)] = \frac{1}{2}(8x+9)^{-1/2}$$

Using chain rule we get,

$$f'(x) = \frac{1}{2}(8x+9)^{-1/2} \cdot 8 = 4(8x+9)^{-1/2} = \frac{4}{\sqrt{8x+9}}$$

(b) Let $g(x) = x^4$ and $h(x) = 12x+9$, then $g'(x) = 4x^3$, $h'(x) = 12$
Note that $f(x) = (12x+9)^4 = (h(x))^4 = g[h(x)]$
Substituting $h(x)$ for x in $g'(x)$,

$$g'[h(x)] = g'[(12x+9)] = 4(12x+9)^3.$$

Using chain rule we get,

$$f'(x) = 4(12x+9)^3 \cdot 12 = 48(12x+9)^3$$

(c) Let $g(x) = 2x^3$ and $h(x) = 4x^2 + 9$, then $g'(x) = 6x^2$, $h'(x) = 8x$
Note that $f(x) = 2(4x^2 + 9)^3 = 2(h(x))^3 = g[h(x)$.
Substituting $h(x)$ for x in $g'(x)$,

$$g'[h(x)] = g'[(4x^2 + 9)] = 6(4x^2 + 9)^2.$$

Using chain rule we get,

$$f'(x) = 6(4x^2 + 9)^2 \cdot 8x = 48x(4x^2 + 9)^2$$

■

The chain rule is also expressed in the following notation. If y is a function of u and u is a function of x, that is $y = f(u)$ and $u = g(x)$, then the derivative of y with respect to x is given by the chain rule as

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

Problem 4.8. Differentiate the following functions:

(a) $y = (13x - 4)^6$

(b) $y = (x^2 + 5)^3$

Solution.

(a) Let $y = u^6$ and $u = 13x - 4$, then

$$\frac{dy}{du} = 6u^5, \qquad \frac{du}{dx} = 13$$

Substitute in the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

we get,

$$\frac{dy}{dx} = 6u^5 \cdot 13 = 78u^5$$

Then replace u with $13x - 4$ we get,

$$\frac{dy}{dx} = 78(13x - 4)^5$$

(b) Let $y = u^3$ and $u = x^2 + 5$, then

$$\frac{dy}{du} = 3u^2, \qquad \frac{du}{dx} = 2x$$

Substitute in the chain rule

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

we get,

$$\frac{dy}{dx} = 3u^2 \cdot 2x = 6xu^2$$

Then replace u with $x^2 + 5$ we get,

$$\frac{dy}{dx} = 6x(x^2 + 5)^2$$

■

4.3 Higher Order Derivatives

The second-order derivative, written $f''(x)$, measures the slope and rate of change of the first derivative. The third-order derivative, written $[f'''(x)]$, measures the slope and rate of change of the second-order derivative, etc. Higher-order derivatives are found by applying the rules of differentiation to lower-order derivatives.

4.4 Higher Order Derivative Notation

Given $y = f(x)$, the following are some commonly used higher-order derivative notations:

First order :	$f'(x)$	y'	$\frac{dy}{dx}$	$\frac{df}{dx}$	$\frac{d}{dx}(y)$	$D_x[f(x)]$
Second order :	$f''(x)$	y''	$\frac{d^2y}{dx^2}$	$\frac{d^2f}{dx^2}$	$\frac{d}{dx}\left(\frac{dy}{dx}\right)$	$D_x^2[f(x)]$
Third order :	$f'''(x)$	y'''	$\frac{d^3y}{dx^3}$	$\frac{d^3f}{dx^3}$	$\frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$	$D_x^3[f(x)]$
Fourth order :	$f^{(4)}(x)$	$y^{(4)}$	$\frac{d^4y}{dx^4}$	$\frac{d^4f}{dx^4}$	$\frac{d}{dx}\left(\frac{d^3f}{dx^3}\right)$	$D_x^4[f(x)]$
n th order :	$f^{(n)}(x)$	$y^{(n)}$	$\frac{d^ny}{dx^n}$	$\frac{d^nf}{dx^n}$	$\frac{d}{dx}\left[\frac{d^{(n-1)}y}{dx^{(n-1)}}\right]$	$D_x^n[f(x)]$

Example 4.4.1. Given $f(x) = 12x^4 - 18x^2 + 5$

$$\begin{aligned} f'(x) &= 48x^3 - 36x \\ f''(x) &= 144x^2 - 36 \\ f'''(x) &= 288x \\ f^{(4)}(x) &= 288 \\ f^{(5)}(x) &= 0 \end{aligned}$$

Problem 4.9. For each of the following functions (1) find the second-order derivative and (2) evaluate it at $x = 3$.

(a) $f(x) = 2x^4 - 5x^2 + 29$

(b) $f(x) = (6x - 5)^3$

(c) $f(x) = (x^3 - 4)(5x^2 + 9)$

Solution.

(a) (1) $f'(x) = 8x^3 - 10x$, $f''(x) = 24x^2 - 10$

(2) At $x = 3$, $\quad f''(3) = 24(3)^2 - 10 = 24(9) - 10 = 216 - 10 = 206$

(b) (1)

$$\begin{aligned} \frac{df}{dx} &= 3(6x-5)^2 \cdot 6 = 18(6x-5)^2 \\ \frac{d^2f}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) &= \frac{d}{dx}[18(6x-5)^2] \\ &= 36(6x-5) \cdot 6 \\ &= 216(6x-5) \\ &= 1296x - 1080 \end{aligned}$$

(2) At $x = 3$, $\quad \dfrac{d^2f}{dx^2} = 1296(3) - 1080 = 3888 - 1080 = 2808$

(c) (1)

$$\begin{aligned}\frac{df}{dx} &= (x^3-4)(10x)+(5x^2+9)(3x^2)\\ &= 10x^4-40x+15x^4+27x^2\\ &= 25x^4+27x^2-40x\\ \frac{d^2f}{dx^2} &= 100x^3+54x-40\end{aligned}$$

(2) At $x=3$, $\dfrac{d^2f}{dx^2}=100(3)^3+54(3)-40=2700+162-40=2822$

■

Problem 4.10. Find the successive derivatives of the function $y=(5x-9)^3$.

Solution.

$$\begin{aligned}y' &= 3(5x-9)^2\cdot 5 = 15(5x-9)^2\\ y'' &= 15\cdot 2(5x-9)\cdot 5 = 150(5x-9) = 750x-1350\\ y''' &= 750\\ y^{(4)} &= 0\end{aligned}$$

■

4.5 Implicit Differentiation

Functions in which x and y are both located on the same side of the equal sign are called *implicit functions.* Functions in which y located on the left of the equal sign and all the x terms are on the right are called *explicit functions.*

Example 4.7. .
$y=2x,\ y=x^2-5x+1,\ y=(x^2-3)(x+5)$ are explicit functions.
$2x+3y=5,\ 2x^3-4xy-2x=25,\ 20x^2y=100$ are implicit functions.

Some implicit functions can be converted easily to explicit functions by solving for y in terms of x; others cannot. For implicit functions and equations which cannot readily be solved for y in terms of x, the derivative dy/dx may be found by means of *implicit differentiation.*

Note that using chain rule we can write

$$\frac{d}{dx}(f(y)) = \frac{d}{dy}(f(y)) \cdot \frac{dy}{dx}.$$

That is for differentiating a function y with respect to x, we differentiate with respect to y and then multiply by $\frac{dy}{dx}$

Example 4.8. Suppose we want to differentiate the implicit function

$$y^2 + x^3 - y^3 + 6 = 3y$$

with respect to x.

We differentiate each term with respect to x:

$$\frac{d}{dx}(y^2) + \frac{d}{dx}(x^3) - \frac{d}{dx}(y^3) + \frac{d}{dx}(6) = \frac{d}{dx}(3y)$$

Differentiating functions of x with respect to x is straightforward. But when differentiating a function of y with respect to x we must remember above rule.

$$\frac{d}{dy}(y^2) \cdot \frac{dy}{dx} + 3x^2 - \frac{d}{dy}(y^3) \cdot \frac{dy}{dx} + 0 = \frac{d}{dy}(3y) \cdot \frac{dy}{dx}$$

$$2y\frac{dy}{dx} + 3x^2 - 3y^2\frac{dy}{dx} = 3\frac{dy}{dx}$$

We rearrange this to collect all terms involving $\frac{dy}{dx}$ together

$$\begin{aligned} 3x^2 &= 3\frac{dy}{dx} - 2y\frac{dy}{dx} + 3y^2\frac{dy}{dx} \\ 3x^2 &= (3 - 2y + 3y^2)\frac{dy}{dx} \\ \frac{dy}{dx} &= \frac{3x^2}{3 - 2y + 3y^2} \end{aligned}$$

This is our expression for $\frac{dy}{dx}$.

4.6 Applications for Business, Economics, and Social Sciences

Problem 4.11. For each of the following total revenue(R), total cost (C), and profit (π) functions, find the derivative, called marginal function and evaluate it at $x = 8$.

(a) $R = 50x - x^2$

(b) $C = x^2 + 10x + 48$

(c) $\pi = 3x^2 - 28x + 132$

Solution.

(a)

$$\begin{aligned} R &= 50x - x^2 \\ \Rightarrow R' &= 50 - 2x \\ R'(8) &= 50 - 2(8) = 34 \end{aligned}$$

(b)

$$\begin{aligned} C &= x^2 + 10x + 48 \\ \Rightarrow C\,' &= 2x + 10 \\ C\,'(8) &= 2(8) + 10 = 26 \end{aligned}$$

(c)

$$\begin{aligned} \pi &= 3x^2 - 28x + 132 \\ \Rightarrow \pi' &= 6x - 28 \\ \pi'(8) &= 6(8) - 28 = 20 \end{aligned}$$

(d)

$$\begin{aligned} \pi &= 3x^2 - 28x + 132 \\ \Rightarrow \pi' &= 6x - 28 \\ \pi'(8) &= 6(8) - 28 = 20 \end{aligned}$$

Problem 4.12. Given that total revenue function

$$R = \frac{600x}{x+3}$$

Find

(a) The marginal revenue function and

(b) The marginal revenue of the seventeenth unit

by means of the quotient rule.

Solution.

(a) The marginal revenue MR $= \dfrac{dR}{dx}$

$$\begin{aligned} R &= \frac{600x}{x+3} \\ \Rightarrow \frac{dR}{dx} &= \frac{(x+3)600 - 600x}{(x+3)^2} \\ &= \frac{600x + 1800 - 600x}{(x+3)^2} = \frac{1800}{(x+3)^2} \end{aligned}$$

(b) The marginal revenue of the seventeenth unit = MR(17)

$$MR(17) = \left.\frac{dR}{dx}\right|_{x=17} = \frac{1800}{[(17)+3]^2} = \frac{1800}{(20)^2} = \frac{1800}{400} = 4.5$$

■

Problem 4.13. Find the marginal revenue function associated with the supply function $P = Q^2 + 4Q + 9$, where P= price, Q=quantity. Evaluate them at $Q = 5$.

Solution. To find a marginal revenue function R', given a supply or demand function, first find the total revenue function $R = P \cdot Q$ and the take the derivative of R with respect to Q.

$$\begin{aligned} R &= P \cdot Q = (Q^2 + 4Q + 9)Q = Q^3 + 4Q^2 + 9Q \\ R' &= 3Q^2 + 8Q + 9 \\ R'(5) &= 3(5)^2 + 8(5) + 9 = 75 + 40 + 9 = 124 \end{aligned}$$

Problem 4.14. Given the average cost function :

$$A = 6Q + 9 + \frac{120}{Q}$$

Find the marginal cost function.

Solution. We first find the total cost function $C = A \cdot Q$; then take the derivative of C with respect to Q to get the marginal cost $C\,'$.

$$\begin{aligned} C &= A \cdot Q = \left(6Q + 9 + \frac{120}{Q}\right) \cdot Q \\ &= 6Q^2 + 9Q + 120 \\ C\,' &= 12Q + 9 \end{aligned}$$

■

Problem 4.15. If the cost function is $C = 8Q + 4\sqrt{Q} + 95$ and the production time schedule is $Q = 150t + 2700$, find the rate of change of cost with respect to time at $t = 6$

Solution.

$$\frac{dC}{dt} = \frac{dC}{dQ} \cdot \frac{dQ}{dt}$$

Here

$$\begin{aligned} C &= 8Q + 4\sqrt{Q} + 95 \\ C &= 8Q + 4Q^{1/2} + 95 \\ \frac{dC}{dQ} &= 8 + 4 \cdot \frac{1}{2} Q^{(1/2)-1} = 8 + 4 \cdot \frac{1}{2} Q^{-1/2} \\ &= 8 + \frac{2}{\sqrt{Q}} \end{aligned}$$

and $\frac{dQ}{dt} = 150$. Hence

$$\frac{dC}{dt} = \left(8 + \frac{2}{\sqrt{Q}}\right) 150 = 1200 + \frac{300}{\sqrt{Q}}$$

$$\left.\frac{dC}{dt}\right|_6 = 1200 + \frac{300}{\sqrt{150(6)+2700}} = 1200 + \frac{300}{\sqrt{900+2700}}$$

$$= 1200 + \frac{300}{\sqrt{3600}} = 1200 + \frac{300}{60} = 1205$$

■

4.7 Exercises

1. Differentiate following functions:

   (a) $y = -27$

   (b) $f(x) = 25 - 6x$

   (c) $y = \frac{1}{8x^3}$

2. Find the derivative of following functions:

   (a) $y = (2x^3 - 4x^2 + 7x)(x^2 - 9)$

   (b) $10\sqrt{x}(6x^2 - 7)$

   (c) $y = \frac{10x^4}{x^2 + 8x + 25}$

   (d) $y = (x^2 - 7x + 4)^3$

   (e) $y = \frac{1}{\sqrt{3x^2 + 7}}$

3. Find the derivative of following functions:

   (a) $y = 3(5x^2 + 11)^4$

   (b) $y = (6x + 8)(4x + 9)^5$

   (c) $y = \frac{4x(3x - 5)}{2x + 1}$

   (d) $y = (4x - 5) \cdot \frac{2x^5}{3x + 2}$

4. For each of the following functions, (1) find the second-order derivative and (2) evaluate it at $x = 3$.

   (a) $y = x^5 - 6x^3 - 4x$

   (b) $y = \frac{x - 1}{x + 1}$

5. For each of the following functions, find the successive derivatives:
   (a) $y = 2x^4 - 4x^3 + 5x^2 + 13x - 27$
   (b) $f(x) = (6x^2 - 4)(5x + 7)$

6. Use implicit differentiation to find the derivative dy/dx for each of the following equations:
   (a) $8x^3 - y^2 = 45$
   (b) $2y^3 - 5y^2 + 7x^5 = 102$
   (c) $(x^3 + 5y)^2 = x^4$

7. Find the marginal revenue function associated with the supply function $P = \frac{1}{2}Q^2 + 3Q + 8$, where P= price, Q=quantity. Evaluate them at $Q = 5$.

8. A projectile shot straight in to the air has height in feet $S(t) = 288t - 16t^2$ after t seconds. Find (a) the velocity $V(t)$ at $t = 3$, (b)the acceleration $A(t)$ at $t = 5$, (c) the time t the object will hit the ground, and (d) the velocity with which it hits the ground.

9. Given the average cost function :

$$A = 2Q^2 - 5Q + 7 + \frac{200}{Q}$$

   Find the marginal cost function.

10. If the cost function is $C = 8Q + 4\sqrt{Q} + 95$ and the production time schedule is $Q = 150t + 2700$, find the rate of change of cost with respect to time at $t = 6$.

Chapter 5

Uses of the Derivative

5.1 Increasing and Decreasing Functions

A function $f(x)$ is said to be *increasing at* $x = a$ if in the area immediately surrounding the point $[a, f(a)]$ the graph of the function rises as it moves from left to right. The function is *decreasing at* $x = a$ if in the area close to $[a, f(a)]$ the graph falls as it moves from left to right.

If the first derivative of a function is positive at $x = a$, therefore, its rate of change at $x = a$ is positive and its slope is positive, so we know the function is increasing at a. Similarly, if the first derivative is negative at $x = a$, we know the function is decreasing at a. In short as seen in Figure 5.1, that is

$$\begin{aligned} f'(a) > 0: &\quad \text{Increasing function at } x = a \\ f'(a) < 0: &\quad \text{Decreasing function at } x = a \end{aligned}$$

Figure 5.1

5.2 Concavity

A function $f(x)$ is said to be *concave upward (or convex)* at $x = a$ if in the area in close proximity to the point $[a, f(a)]$, the graph of the function lies completely above its tangent line. A function is *concave downward* at $x = a$ if in the area immediately around the point $[a, f(a)]$, the graph lies completely below its tangent line.

If the second derivative of a function is positive at $x = a$, the function is concave upward at $x = a$. If the second derivative is negative at $x = a$, the function is concave downward at a. The slope of the first derivative is irrelevant for concavity.

$$f''(a) > 0 : \qquad \text{Concave upward at } x = a$$
$$f''(a) < 0 : \qquad \text{Concave downward at } x = a$$

. Concave Upward at $x = a$

. Concave Downward at $x = a$

Figure 5.2

Illustrations of the different combinations of slopes and concavity are presented in Figure 5.2 and summarized in Table 5.1

Figure	**Derivative**	**Function**
Figure 5.2 (a)	$f'(a) > 0$ $f''(a) > 0$	$f(x)$ increasing $f(x)$ concave upward
Figure 5.2 (b)	$f'(a) < 0$ $f''(a) > 0$	$f(x)$ decreasing $f(x)$ concave upward
Figure 5.2 (c)	$f'(a) > 0$ $f''(a) < 0$	$f(x)$ increasing $f(x)$ concave downward
Figure 5.2 (d)	$f'(a) < 0$ $f''(a) < 0$	$f(x)$ decreasing $f(x)$ concave downward

Table 5.1

Problem 5.1. Determine whether the following functions are increasing, decreasing, or stationary at $x = 3$:

(a) $y = 125 - 9x$

(b) $y = 3x^2 - 25$

(c) $y = x^3 - 5x^2 + 3x - 54$

(d) $y = 2x^2 - 48x + 27$

Solution.

(a)

$$\begin{aligned} y &= 125 - 9x \\ y' &= -9 \\ y'(3) &= -9 < 0 \end{aligned}$$

So the function $y = 125 - 9x$ is decreasing at $x = 3$.

(b)

$$\begin{aligned} y &= 3x^2 - 25 \\ y' &= 6x \\ y'(3) &= 6(3) = 18 > 0 \end{aligned}$$

So the function $y = 3x^2 - 25$ is increasing at $x = 3$.

(c)

$$\begin{aligned} y &= x^3 - 5x^2 + 3x - 54 \\ y' &= 3x^2 - 10x + 3 \\ y'(3) &= 3(3)^2 - 10(3) + 3 \\ &= 27 - 30 + 3 = 0 \end{aligned}$$

So the function $y = x^3 - 5x^2 + 3x - 54$ is stationary at $x = 3$.

(d)

$$\begin{aligned} y &= 2x^2 - 48x + 27 \\ y' &= 4x - 48 \\ y'(3) &= 4(3) - 48 \\ &= 12 - 48 = -36 < 0 \end{aligned}$$

So the function $y = 2x^2 - 48x + 27$ is decreasing at $x = 3$. ■

Problem 5.2. Test to see if the following functions are concave upward or concave down ward at $x = 5$:

(a) $y = x^2 + 12x + 11$

(b) $y = -4x^3 + 5x^2 + 14x - 15$

(c) $y = 3x^3 - 7x^2 - 8x + 93$

Solution.

(a)

$$\begin{aligned} y &= x^2 + 12x + 11 \\ y' &= 2x + 12 \\ y'' &= 2 \\ y''(5) &= 2 > 0 \end{aligned}$$

So the function $y = x^2 + 12x + 11$ is concave upward at $x = 5$.

(b)

$$\begin{aligned} y &= -4x^3 + 5x^2 + 14x - 15 \\ y' &= -12x^2 + 10x + 14 \\ y'' &= -24x + 10 \\ y''(5) &= -24(5) + 10 = -110 < 0 \end{aligned}$$

So the function $y = -4x^3 + 5x^2 + 14x - 15$ is concave downward at $x = 5$.

(c)

$$\begin{aligned} y &= 3x^3 - 7x^2 - 8x + 93 \\ y' &= 9x^2 - 14x \\ y'' &= 18x - 14 \\ y''(5) &= 18(5) - 14 = 76 > 0 \end{aligned}$$

So the function $y = 3x^3 - 7x^2 - 8x + 93$ is concave upward at $x = 5$. ■

5.3 Extreme Points

An extreme point of a function is a point where the function is at a relative maximum or minimum. For a function to be at a relative maximum or minimum at $x = a$, the function must be neither increasing nor decreasing at a. we know that for a function to be neither increasing nor decreasing at $x = a$, its first derivative must equal zero or be undefined at a. A point where the derivative equals zero or is undefined is called a *critical point*.

To distinguish between a relative maximum and a relative minimum, the second derivative is used. Assuming that a is a critical point.

1. If $f''(a) > 0$, which indicates that the function is concave upward and the graph of the function lies completely above its tangent line at $x = a$, the function must be a relative minimum at $x = a$.

2. If $f''(a) < 0$, which indicates that the function is concave downward and the graph of the function lies completely below its tangent line at $x = a$, the function must be a relative minimum at $x = a$.

3. If $f''(a) = 0$, the test is inconclusive.

For functions which are differentiable at all values of x in their domains, which are called differentiable or smooth functions and assumed in a text of this nature, one need only consider the case where $f'(x) = 0$ in searching for critical points. In short,

$$f'(a) = 0, f''(a) > 0: \quad \text{Relative minimum at } x = a$$
$$f'(a) = 0, f''(a) < 0: \quad \text{Relative maximum at } x = a$$

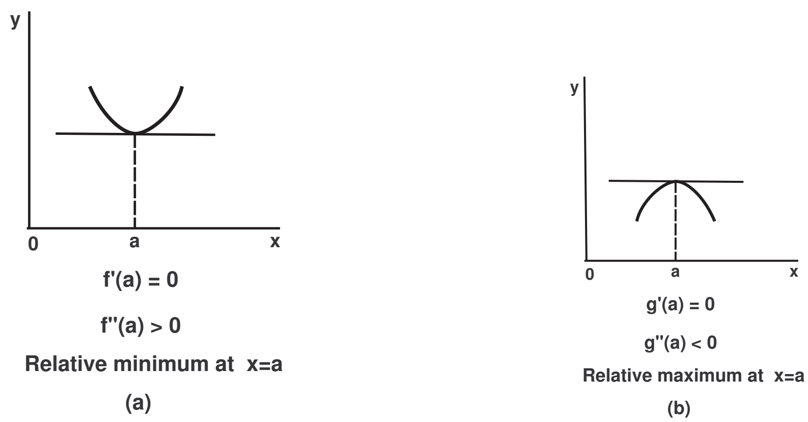

Figure 5.3

Example 5.1. In Figure 5.3(a), $f'(a) = 0$, indicating that the function is at a relative plateau at $x = a$. With $f''(a) > 0$, $f(x)$ is concave upward with the graph everywhere above its tangent line at $x = a$. Hence $f(x)$ must be at a relative minimum at $x = a$.

In Figure 5.3(b), with $g'(a) = 0$, $g(x)$ is at a relative plateau at $x = a$, $g''(a) < 0$, indicating that $g(x)$ is concave downward and the graph everywhere below its tangent line at $x = a$. Hence $g(x)$ must be at a relative maximum at $x = a$.

5.4 Optimization

Optimization is the process of finding the relative maximum or minimum of a function, also called the *relative extrema*. Without resorting to a graph, this can be done in two easy steps. Assuming the usual differentiable function,

1. Take the first derivative, set it equal to zero, and solve for x_0. This step is frequently referred to as the *first-order condition* and x_0 is called a critical point or a critical value of x.

2. Take the second derivative, evaluate it at the critical point(s), and check the sign(s). If at a critical point,

$$
\begin{aligned}
f''(x_0) &> 0 : \quad && \textit{Concave upward , relative minimum} \\
f''(x_0) &< 0 : \quad && \textit{Concave downward , relative minimum} \\
f''(x_0) &= 0 : \quad && \textit{The test is inconclusive}
\end{aligned}
$$

This step is called the *second-order condition* or *second-derivative test*.

Example 5.2. To optimize $y = 3x^3 - 36x^2 + 135x - 17$, simply test the first and second-order conditions, as outlined above:

(a) Take the first derivative, set it equal to zero, and solve for x_0 to find the critical values.

$$
\begin{aligned}
y &= 3x^3 - 36x^2 + 135x - 17 \\
y' &= 9x^2 - 72x + 135 \\
&= 9(x^2 - 8x + 15) \\
&= 9(x - 3)(x - 5)
\end{aligned}
$$

$y'(x) = 0 \quad \Rightarrow 9(x-3)(x-5) = 0 \quad \Rightarrow (x-3) = 0 \; \textit{or} \; (x-5) = 0$
So $x_0 = 3$ and $x_0 = 5$ are critical values.

(b) Take the second derivative,

$$y'' = 18x - 72$$

evaluate it at the critical values, and test the second-order condition by checking the signs for concavity to distinguish between a relative maximum and minimum.

$$
\begin{aligned}
f''(3) &= 18(3) - 72 = -18 < 0 \quad && \text{Concave down, relative maximum} \\
f''(5) &= 18(5) - 72 = 18 > 0 \quad && \text{Concave up, relative minimum}
\end{aligned}
$$

Hence the curve is concave downward and has a relative maximum at $x = 3$, also the curve is concave upward and has a relative minimum at $x = 5$

Problem 5.3. For each of the following functions (1) find the critical value(s) and (2) determine whether at the critical value(s) the function is at a relative maximum or minimum :

(a) $f(x) = 3x^2 - 42x + 34$

(b) $f(x) = 2x^3 - 24x^2 + 72x - 15$

(c) $\dfrac{x^2+9}{x} \qquad (x \neq 0)$

Solution.

(a) (1) Take the first derivative, set it equal to zero, and solve for x_0 to find the critical value(s).
$f'(x) = 6x - 42 = 0 \quad \Rightarrow x_0 = 7$, which is the critical value.
(2) Take the second derivative, evaluate it at the critical value(s), and check for concavity to distinguish between a relative maximum and a relative minimum.
$f''(x) = 6 \qquad \Rightarrow f''(7) = 6 > 0$
Therefore the function is concave up at $x = 7$, so the function has a relative minimum at $x = 7$.

(b) (1) Here $f'(x) = 6x^2 - 48x + 72 \qquad f'(x) = 0$ implies

$$\begin{aligned} 6x^2 - 48x + 72 &= 0 \\ 6(x^2 - 8x + 12) &= 0 \\ 6(x-2)(x-6) &= 0 \end{aligned}$$

This is possible if $(x-2) = 0$ or $(x-6) = 0$, that is if $x = 2$ or $x = 6$.
So $x_0 = 2$ and $x_0 = 6$ are critical values.
(2) Here $f''(x) = 12x - 48$. Now we evaluate $f''(x)$ at the critical values. Note that,

$$f''(2) = 12(2) - 48 = -24 < 0$$

So the function is concave down and has a relative maximum at $x = 2$.

$$f''(6) = 12(6) - 48 = 24 > 0$$

So the function is concave up and has a relative minimum at $x = 6$.

(c) (1)

$$f'(x) = \frac{x(2x) - (x^2+9)(1)}{x^2} = \frac{x^2-9}{x^2}$$

$$f'(x) = 0 \qquad \Rightarrow \frac{x^2-9}{x^2} = 0 \qquad \Rightarrow x^2 - 9 = 0 \Rightarrow x = \pm 3$$

So the critical values are $x_0 = 3$ and $x_0 = -3$.

(2)

$$f''(x) = \frac{x^2(2x) - (x^2-9)(2x)}{(x^2)^2} = \frac{18x}{x^4} = \frac{18}{x^3}$$

Note that

$$f''(3) = \frac{18}{(3)^3} = \frac{2}{3} > 0$$

So the function is concave up and has a relative minimum at $x = 3$.

$$f''(-3) = \frac{18}{(-3)^3} = -\frac{2}{3} < 0$$

So the function is concave down and has a relative maximum at $x = -3$.

■

5.5 Inflection Points

An *inflection point* is a point on the graph where the function changes from concave upward to concave downward or vice versa. Between any adjoining maximum and minimum or minimum and maximum there must be an inflection point where the function changes concavity.

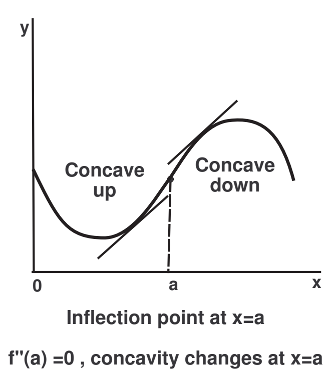

Figure 5.4

Inflection points can occur only where the second derivative equals zero or is undefined. The sign of the first derivative is immaterial for an inflection point. In sum, for an inflection point a

1. $f''(a) = 0$ or is undefined.
2. Concavity changes at $x = a$

Note 5.1. For differentiable functions, which have smooth graphs and continuous derivatives, one need only consider the case where $f''(x) = 0$ in step 1. Such functions are assumed in this text and include all polynomial functions and all rational functions except where undefined.

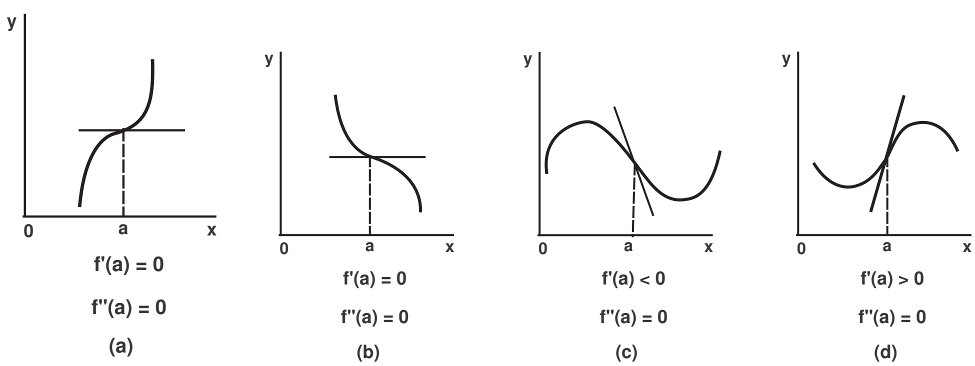

Figure 5.5

Example 5.3. An inflection point can also be identified as a point on the graph where the graph crosses its tangent line. Different possibilities for inflection points are given in Figure 5.5. Note that at every inflection point $[a, f(a)]$ (1) the function changes concavity and (2) the graph crosses its tangent line. Note, too, that while the sign of the first derivative can be zero, negative, or positive, the second derivative either equals zero at a point of inflection or is undefined.

Note 5.2. The second-derivative test would be inconclusive for functions such as those illustrated in Figure 5.5 (a) and (b) where at the critical values we have inflection points.

In the event that $f''(x_0) = 0$ and without a graph for clarification, one may simply test for change of concavity by evaluating the second derivative at points slightly to the left and right of the critical value or take successive derivatives.

In the latter case, if the first non zero value of a higher-order derivative evaluated at the critical value is an odd-numbered derivative (3rd,5th etc.), the function is at an inflection point; if the first non zero value of a higher-order derivative evaluated at the critical value is an even-numbered derivative, the function is at an extreme point, with a positive value of the derivative signifying a relative minimum and a negative value indicating a relative maximum.

Problem 5.4. For the following functions (1) find the critical values and (2) test to see if at the critical values the function is at a relataive minimaum, maximum, or inflection point:

(a) $y = (5 - x)^4$

(b) $y = (x - 8)^3$

Solution.

(a) (1) Take the first derivative, set it equal to zero, and solve for x_0 to find the critical value(s)
$y' = 4(5 - x)^3(-1) = -4(5 - x)^3 = 0 \quad \Rightarrow x_0 = 5$ which is the critical value.
(2) Taking the second derivative, evaluate it at the critical value(s), and checking the sign for concavity to distinguish between a relative maximum and a relative minimum or a possible inflection point
$y'' = -12(5 - x)^2(-1) = 12(5 - x)^2 \qquad \Rightarrow y''(5) = 12(5 - 5)^2 = 0$
Therefore the test is inconclusive .
When the second derivative test is inconclusive, continue to take successively higher derivatives and evaluate them at the critical values until we come to the first higher-order derivative that is non zero:

$$\begin{aligned} y''' &= 24(5 - x)(-1) = -24(5 - x) \\ y'''(5) &= -24(5 - 5) = 0 \qquad \textit{Test is inconclusive} \\ y^{(4)} &= 24 \\ y^{(4)}(5) &= 24 \end{aligned}$$

With the first non-zero higher order derivative an even-numbered derivative, $y(5)$ is an extreme point; with $y^{(4)}(5) = 24 > 0$, y is concave upward and at a relative minimum at $x = 5$.

(b) (1) Here $y' = 3(x-8)^2$, $y' = 0 \Rightarrow 3(x-8)^2 = 0$ $\Rightarrow x_0 = 8$
So the critical value is $x_0 = 8$.
(2) Here $y'' = 6(x-8)$ $y''(8) = 6(8-8) = 0$ *Test is inconclusive*
Continuing to take successively higher derivatives and evaluate them at the critical value in search of the first higher-order derivative that is does not equal to zero:

$$y''' = 6$$
$$y'''(8) = 6 > 0$$

With the first nonzero higher order derivative an odd-numbered derivative, y is at an inflection and not at an extreme point.

■

5.6 Curve Sketching

With the information provided by the first and the second derivatives one can readily determine the overall shape of the graph of a function and do a rough sketch of the graph in five easy steps. Given a differentiable function $f(x)$

1. Take the first derivative $f'(x)$ to get a rough idea of where the function is increasing and decreasing.
2. Find the extreme points of $f(x)$ by setting $f'(x)$ equal to zero and solving for x_0. Then evaluate $f(x)$ at x_0.
3. Take the second derivative, evaluate it at x_0, and check the sign to determine concavity in order to distinguish between a relative maximum, minimum, or inflection point.
4. Look for inflection points where $f''(x) = 0$ and where concavity changes.
5. Determine the intercepts, if helpful, and draw the graph.

The following table gives the shape of the graph of f corresponding to four cases determined by the signs of f' and f''. For example, first row first column corresponds to that both f' and f'' are positive: the figure indicates that the graph goes up and bends up.

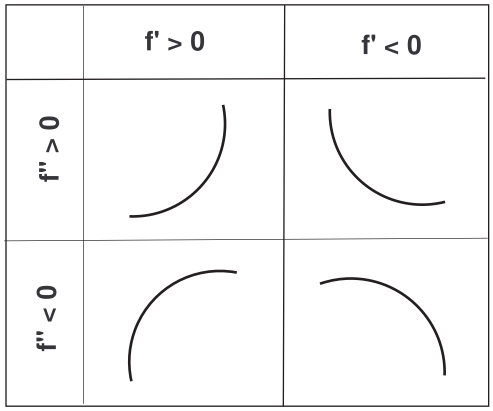

Table 5.2

Problem 5.5. For each of the following functions (1) find the critical values, (2)test for concavity to determine relative maxima or minima, (3) check for any inflection point, (4) evaluate the function at the critical values, and (5) sketch the graph of the function.

(a) $f(x) = -x^2 + 12x - 28$

(b) $f(x) = -x^3 - 3x^2 + 24x + 32$

(c) $f(x) = -(x-4)^3$

Solution.

(a) (1) $f'(x) = -2x + 12, \qquad f'(x) = 0 \qquad \Rightarrow -2x + 12 = 0 \quad \Rightarrow x_0 = 6$
Hence the critical value is $x_0 = 6$.

(2) $f''(x) = -2, \qquad f''(6) = -2 < 0$. therefore the curve is concave down and has a relative maximum at $x_0 = 6$.

(3) $f''(x) \neq 0$. So the curve has no inflection point.

(4) $f(6) = -(6)^2 + 12(6) - 28 = -36 + 72 - 28 = 8$. Hence the curve has a relative maximum at $(6, 8)$.

(5) See Figure 5.6.

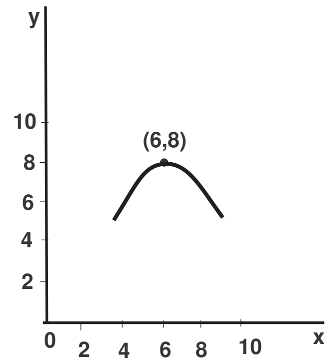

Figure 5.6

(b) (1) $f'(x) = -3x^2 - 6x + 24, \qquad f'(x) = 0$ implies,

$$\begin{aligned} -3x^2 - 6x + 24 &= 0 \\ -3(x^2 + 2x - 8) &= 0 \\ -3(x-2)(x+4) &= 0 \end{aligned}$$

This is possible if $(x - 2)$ or $(x + 4) = 0$, that is if $x = 2$ or $x = -4$ Hence the critical values are $x_0 = 2$ and $x_0 = -4$.

(2) $f''(x) = -6x - 6$

$$f''(2) = -6(2) - 6 = -18 < 0$$

Therefore the curve is concave down and has a relative maximum at $x_0 = 2$.

$$f''(-4) = -6(-4) - 6 = 18 > 0$$

Therefore the curve is concave up and has a relative minimum at $x_0 = -4$.

(3) $f''(x) = 0 \quad \Rightarrow -6x - 6 = 0 \quad \Rightarrow x = -1$.
Note that $f(-1) = 6$, So $(-1, 6)$ is an inflection point.

(4) $f(2) = 60$ and $f(-4) = -48$. Hence the curve has a relative maximum at $(2, 60)$ and relative minimum at $(-4, -48)$.

(5) See Figure 5.7.

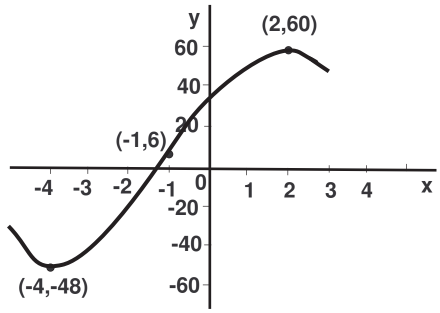

Figure 5.7

(c) (1) $f'(x) = -3(x-4)^2, \qquad f'(x) = 0 \quad \Rightarrow -3(x-4)^2 = 0 \quad \Rightarrow x_0 = 4$

Hence the critical value is $x_0 = 4$.

(2)

$$\begin{aligned} f''(x) &= -6(x-4) \\ f''(4) &= -6(4-4) = 0 \end{aligned}$$

So the test is inconclusive. Continuing on to successively higher-order derivatives in search of the first non zero derivatives,

$$\begin{aligned} f''' &= -6 \\ f'''(4) &= -6 \end{aligned}$$

Note that $f(4) = 0$. So $(4, 0)$ is an inflection point.

(3) and (4) Now we test concavity to the left ($x = 3$) and right ($x = 5$) of $x = 4$ to see which way the graph of the function curves:

Note that,

$$f''(3) = -6(3-4) = 6 > 0$$

Hence the curve is concave upward at $x = 3$.

$$f''(5) = -6(5-4) = -6 < 0$$

So the curve is concave downward at $x = 5$.

(5) Try yourself to draw the curve.

■

Problem 5.6. Find the level of output at which profit π is maximized, given that the total revenue $R = 6400Q - 20Q^2$ and total cost $C = Q^3 - 5Q^2 + 400Q + 52,000$ assume $Q > 0$.

Solution.
(1) Set up the profit function: $\pi = R - C$

$$\begin{aligned} \pi &= 6400Q - 20Q^2 - (Q^3 - 5Q^2 + 400Q + 52,000) \\ &= 6400Q - 20Q^2 - Q^3 + 5Q^2 - 400Q - 52,000 \\ &= -Q^3 - 15Q^2 + 6000Q - 52,000 \end{aligned}$$

(2) Take the first derivative, set it equal to zero, and solve for Q_0 to find the critical values. $\pi' = -3Q^2 - 30Q + 6000$, $\quad \pi' = 0$ implies,

$$\begin{aligned} -3Q^2 - 30Q + 6000 &= 0 \\ -3(Q^2 + 10Q - 2000) &= 0 \\ -3(Q + 50)(Q - 40) &= 0 \end{aligned}$$

This is possible if $(Q + 50) = 0$ or $(Q - 40) = 0$, That is if $Q = -50$ or $Q = 40$
Hence the critical values are $Q_0 = -50$ and $Q_0 = 40$.
(3) Test the second order conditions

$$\pi'' = -6Q - 30$$

we ignore the negative critical value. Note that,

$$\pi''(40) = -6(40) - 30 = -270 < 0$$

Curve is concave down, and has a relative maximum at $Q_0 = 40$. So the profit is maximized at $Q = 40$ where,

$$\pi(40) = -(40)^3 - 15(40)^2 + 6000(40) - 52,000 = 100,000$$

■

5.7 Exercises

1. Determine whether the following functions are increasing, decreasing, or stationary at $x = 3$:
   (a) $y = (4x - 5)^2$ (b)$y = \dfrac{7x - 9}{2x}$ $(x \neq 0)$

2. Test to see if the following functions are concave upward or concave downward at $x = 5$:
   (a) $y = 27 + 13x - 3x^2$ (b) $y = -x(x - 10)^2$ (c) $y = \dfrac{-4}{x - 9}$ $(x \neq 9)$

3. For each of the following functions (1) find the critical value(s) and (2) determine whether at the critical value(s) the function is at a relative maximum or minimum :
   (a) $f(x) = -5x^2 + 78x - 49$ (b) $f(x) = x^4 - 72x^2 + 7$

4. For each of the following functions (1) find the critical values (2)test for concavity to determine relative maxima or minima (3) check for any inflection point (4) evaluate the function at the critical values and (5) sketch the graph of the function.
   (a) $f(x) = 4x^2 - 24x + 40$ (b) $f(x) = -(x - 3)^4$
   (c) $f(x) = 2x^3 - 54x^2 + 480x - 1300$

5. Find the critical value(s) at which each of the follwoing function is optimized and test the second-order condition to distinguish between relative maximum and minimum:
   (a) $f(x) = 6x^2 - 36x + 25$ (b) $f(x) = -3x^2 - 12x + 57$
   (c) $f(x) = 2x^3 + 12x^2 - 192x - 45$ (d) $-4x^2 + 72x - 15$

6. The concentration in milligrams per cubic centimeter of drug A in a person's bloodstream after t hours is given by

   $$C(t) = \frac{0.18t}{t^2 + 3t + 25}$$

   Find when the concentration is strongest.

Chapter 6

Exponential and Logarithmic Functions

6.1 Exponential Functions

Earlier we dealt with power functions such as $y = x^a$, which are composed of a variable base x and a constant exponent a. In this chapter we encounter a new function that is composed of a constant base a and a variable exponent x. It is called an *exponential function* and is defined as

$$y = a^x \qquad a > 0 \quad and \quad a \neq 1$$

For example $f(x) = 2^x$ is an exponential function and $f(x) = \left(\frac{3}{4}\right)^x$ is also an exponential function.

There is a big difference between an exponential function and a polynomial. The function $p(x) = x^3$ is a polynomial. Here the variable, x, is being raised to some constant power. The function $f(x) = 3^x$ is an exponential function; the variable is the exponent. Exponential functions are used to express growth and decay.

The following are some general characteristics of exponential functions.

Given $y = a^x,\ a > 0,\ a \neq 1$:

(a) The domain of the function is the set of all real numbers; the range of the function is the set of all positive real numbers.

(b) For $a > 1$, the function is increasing and concave upward; for $0 < a < 1$, the function is decreasing and concave upward.

(c) at $x = 0, y = 1$, independently of the base.

6.2 Logarithmic Functions

By interchanging the variables of an exponential function f defined by $y = a^x$, it is possible to obtain a new function g defined by $x = a^y$ such that any ordered pair numbers in f will also be found in g in reverse order. For example if $f(2) = 4$, then $g(4) = 2$; $f(3) = 8$, then $g(8) = 3$. The new function g, the inverse of exponential function f is called a *logarithmic function with base a*. Instead of $x = a^y$, the logarithmic function with base a is more commonly written

$$y = \log_a x \qquad a > 0, a \neq 1$$

That is $y = \log_a x$ if and only if $a^y = x$.

Example 6.1. A graph of two functions f and g in which x and y are interchanged, such as $y = 2^x$ and $x = 2^y$ in Figure 6.1 reveals that one function is a *mirror image* of the other along the 45^0 line $y = x$, such that $f(x) = y$, then $g(y) = x$ Note that $x = 2^y$ is equivalent to $y = \log_2 x$.

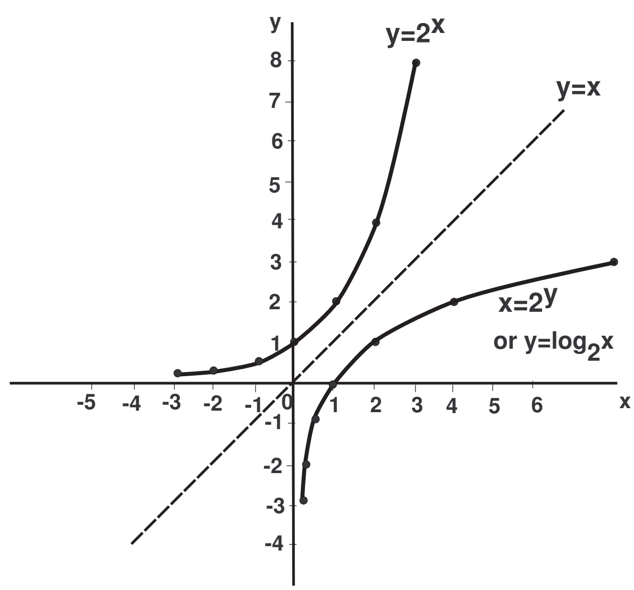

Figure 6.1

The following are some general characteristics of logarithmic functions.
Given $f(x) = \log_a x,\ a > 0,\ a \neq 1$:

(a) The domain of the function is the set of all positive real numbers; the range is the set of all real numbers .

(b) For base $a > 1$, $f(x)$ is increasing and concave down. For $0 < a < 1$, $f(x)$ is decreasing and concave down.

(c) At $x = 1, y = 0$, independently of the base.

Example 6.2. .

(a) $\log_2 16 = 4$ is equivalent to $16 = 2^4$.

(b) $\log_{25} 5 = \frac{1}{2}$ is equivalent to $5 = 25^{1/2}4$.

(c) $\log_2 \frac{1}{8} = -3$ is equivalent to $\frac{1}{8} = 2^{-3}$.

(d) $36 = 6^2$ is equivalent to $\log_6 36 = 2$.

(e) $9 = \sqrt{81}$ is equivalent to $\log_{81} 9 = \frac{1}{2}$.

(f) $\frac{1}{9} = 3^{-2}$ is equivalent to $\log_3 \frac{1}{9} = -2$.

6.3 Properties of Exponents and Logarithms

Assuming $a, b > 0$; $a, b \neq 1$, and x and y any real number,

(a) $a^x \cdot a^y = a^{x+y}$

(b) $a^{-x} = \dfrac{1}{a^x}$

(c) $\dfrac{a^x}{a^y} = a^{x-y}$

(d) $(a^x)^y = a^{xy}$

(e) $a^x \cdot b^x = (ab)^x$

(f) $a^{-x} = \dfrac{1}{a^x}$

(g) $\dfrac{a^x}{b^x} = \left(\dfrac{a}{b}\right)^x$

Letting x and y be any positive real numbers, n any real number, a any positive real number, and $a \neq 1$,

(a) $\log_a(xy) = \log_a x + \log_a y$

(b) $\log_a \frac{x}{y} = \log_a x - \log_a y$

(c) $\log_a x^n = n \log_a x$

(d) $\log_a(\sqrt[n]{x}) = \frac{1}{n}(\log_a x)$

Problem 6.1. Change the following logarithms to their equivalent exponential forms :

(a) $\log_6 36 = 2$

(b) $\log_3 27 = 3$

(c) $\log_2 \frac{1}{32} = -5$

(d) $\log_{81} 3 = \frac{1}{4}$

Solution.

(a) $36 = 6^2$

(b) $27 = 3^3$

(c) $\frac{1}{32} = 2^{-5}$

(d) $3 = 81^{1/4}$

■

Problem 6.2. Change the following exponential forms to logarithmic forms :

(a) $49 = 7^2$

(b) $\frac{1}{8} = 2^{-3}$

(c) $4 = 64^{1/3}$

(d) $27 = 9^{3/2}$

Solution.

(a) $\log_7 49 = 2$

(b) $\log_2 \frac{1}{8} = -3$

(c) $\log_{64} 4 = \frac{1}{3}$

(d) $\log_9 27 = \frac{3}{2}$

■

6.4 Natural Exponential and Logarithmic Functions

The most commonly used base for exponential and logarithmic functions is the irrational number e. e is defined as

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

This limit when rounded to five decimal places equals 2.71828.

Exponential functions to base e are called *natural exponential functions* and are written $y = e^x$; logarithmic functions to base e are called *natural logarithmic functions* and are expressed as $y = \log_e x = \ln x$. Natural exponential and logarithmic functions have the same general properties as other exponential and logarithmic functions.

Problem 6.3. Convert the following natural logarithms into natural exponential functions :

(a) $\ln 24 = 3.17805$

(b) $\ln 0.75 = -0.28768$

(c) $\ln 10 = 2.30258$

Solution.

(a) $24 = e^{3.17805}$

(b) $0.75 = e^{-0.28768}$

(c) $10 = e^{2.30258}$

■

Problem 6.4. Convert the following natural exponential expressions in to equivalent natural logarithmic forms :

(a) $1.5 = e^{0.40547}$

(b) $25 = e^{3.21888}$

Solution.

(a) $\ln 1.5 = 0.40547$

(b) $\ln 25 = 3.21888$

■

Problem 6.5. Solve the following for x, y, or a by finding the equivalent expression :

(a) $y = \log_{20} 400$

(b) $\log_5 x = 3$

(c) $\log_a 4 = \frac{2}{3}$

Solution.

(a) $400 = 20^y \quad \Rightarrow y = 2$

(b) $x = 5^3 = 125$

(c) $4 = a^{2/3} \quad \Rightarrow a = 4^{3/2} = (4^3)^{1/2} = (64)^{1/2} = 8$

■

Problem 6.6. Use the properties of logarithms to write the following expressions as sums, differences, or products :

(a) $\log_a 15x$

(b) $\log_a 28x^3$

(c) $\log_a x^3y^4$

(d) $\log_a \frac{8x}{9y}$

(e) $\log_a \frac{x^5}{y^3}$

Solution.

(a) $\log_a 15x = \log_a 15 + \log_a x$

(b) $\log_a 28x^3 = \log_a 28 + 3\log_a x$

(c) $\log_a x^3y^4 = 3\log_a x + 4\log_a y$

(d) $\log_a \frac{x^5}{y^3} = \log_a x^5 - \log_a y^3 = 5\log_a x - 3\log_a y$

■

Problem 6.7. Use the properties of logarithms to write the following natural logarithmic forms as sums, differences, or products :

(a) $\ln 45x^2$

(b) $\ln x^3y^4$

(c) $\ln(\sqrt[5]{x})$

(d) $\ln \frac{8x}{9y}$

(e) $\ln \sqrt{\frac{x^5}{y^3}}$

Solution.

(a) $\ln 45x^2 = \ln 45 + 2\ln x$

(b) $\ln x^3y^4 = 3\ln x + 4\ln y$

(c) $\ln(\sqrt[5]{x}) = \ln(x)^{1/5} = \frac{1}{5}\ln x$

(d) $\ln \frac{8x}{9y} = \ln 8x - \ln 9y = \ln 8 + \ln x - (\ln 9 + \ln y) = \ln 8 + \ln x - \ln 9 - \ln y$

(e) $\ln\sqrt{\dfrac{x^5}{y^3}} = \ln\left(\dfrac{x^5}{y^3}\right)^{1/2} = \dfrac{1}{2}\ln\dfrac{x^5}{y^3} = \dfrac{1}{2}(\ln x^5 - \ln y^3) = \dfrac{1}{2}(5\ln x - 3\ln y)$

■

Problem 6.8. Use the properties of exponents to simplify the following exponential expressions, assuming $a, b > 0$ and $a \neq b$:

(a) $a^x \cdot a^y$

(b) $a^{2x} \cdot a^{3y}$

(c) $\dfrac{a^{6x}}{a^{4y}}$

(d) $\dfrac{a^x}{b^y}$

(e) $\sqrt{a^{5x}}$

(f) $(a^x)^{5y}$

Solution.
(a) $a^x \cdot a^y = a^{x+y}$ (b) $a^{2x} \cdot a^{3y} = a^{2x+3y}$ (c) $\dfrac{a^{6x}}{a^{4y}} = a^{6x-4y}$

(d) $\dfrac{a^x}{b^y} = \left(\dfrac{a}{b}\right)^x$ (e) $\sqrt{a^{5x}} = (a^{5x})^{1/2} = a^{(5/2)x}$ (f) $(a^x)^{5y} = a^{5xy}$
. ■

Problem 6.9. Simplify the following natural exponential expressions:

(a) $e^{7x} \cdot e^y$ (b) $(e^{4x})^3$ (c) $\dfrac{e^{7x}}{e^{4x}}$ (d) $\dfrac{e^{3x}}{e^{7x}}$

Solution.
(a) $e^{7x} \cdot e^y = e^{7x+y}$ (b) $(e^{4x})^3 = e^{12x}$ (c) $\dfrac{e^{7x}}{e^{4x}} = e^{7x-4x} = e^{3x}$

(d) $\dfrac{e^{3x}}{e^{7x}} = e^{3x-7x} = e^{-4x} = \dfrac{1}{e^{4x}}$
. ■

Problem 6.10. Simplify the following natural logarithmic expressions:

(a) $\ln 9 + \ln x$ (b) $\ln x^6 - \ln x^2$ (c) $\ln 5 - \ln x + \ln 3$ (d) $3\ln\frac{1}{4}$

(e) $\frac{1}{2}\ln 49$ (f) $2\ln 6 - \frac{1}{2}\ln 16$

Solution.
(a) $\ln 9 + \ln x = \ln 9x$

(b) $\ln x^6 - \ln x^2 = \ln \dfrac{x^6}{x^2} = \ln x^4$

(c) $\ln 5 - \ln x + \ln 3 = \ln 5 + \ln 3 - \ln x = \ln 5 \cdot 3 - \ln x = \ln \dfrac{5.3}{x} = \ln \dfrac{15}{x}$

(d) $3 \ln \frac{1}{4} = \left(\ln \frac{1}{4}\right)^3$

(e) $\frac{1}{2} \ln 49 = \ln 49^{1/2} = \ln 7$

(f) $2 \ln 6 - \frac{1}{2} \ln 16 = \ln 6^2 - \ln 16^{1/2} = \ln 36 - \ln 4 = \ln \dfrac{36}{4} = \ln 9$
. ■

6.5 Solving Natural Exponential and Logarithmic Functions

Natural exponential functions and natural logarithmic functions are inverses of each other. Note that

$$e^{\ln a} = a \qquad e^{\ln x} = x \qquad e^{\ln f(x)} = f(x)$$

Conversely,

$$\ln e^a = a \qquad \ln e^x = x \qquad \ln e^{f(x)} = f(x)$$

Problem 6.11. Simplify each of the following exponential expressions:

(a) $e^{5 \ln x}$ (b) $e^{3 \ln x + 2 \ln y}$ (c) $e^{(1/2) \ln x}$ (d) $e^{5 \ln x - 3 \ln y}$

Solution.
(a) $e^{5 \ln x} = e^{\ln x^5} = x^5$

(b) $e^{3 \ln x + 2 \ln y} = e^{\ln x^3 + \ln y^2} = e^{\ln x^3} \cdot e^{\ln y^2} = x^3 y^2$

(c) $e^{(1/2)\ln x} = e^{\ln x^{1/2}} = x^{1/2} = \sqrt{x}$

(d) $e^{5\ln x - 3\ln y} = e^{5\ln x} \cdot e^{-3\ln y} = e^{\ln x^5} \cdot e^{-\ln y^3} = \dfrac{e^{\ln x^5}}{e^{\ln y^3}} = \dfrac{x^5}{y^3}$. ■

Problem 6.12. Simplify each of the following logarithmic expressions:

(a) $\ln e^x$ (b) $\ln e^{x^2}$ (c) $5\ln e^{3x^2+4y}$

Solution.
(a) $\ln e^x = x$ (b) $\ln e^{x^2} = x^2$ (c) $5\ln e^{3x^2+4y} = 5(3x^2+4y) = 15x^2+20y$
. ■

Problem 6.13. Solve the following equations for x:

(a) $4\ln x - 10 = 0$ (b) $3e^{x-4} = 24$ (c) $2e^{3x} = 3616$ (d) $\frac{1}{3}e^{x^2} = 2701$

Solution.

(a) (1) First solve algebraically for $\ln x$,

$$\begin{aligned} 4\ln x - 10 &= 0 \\ 4\ln x &= 10 \\ \ln x &= \frac{10}{4} \\ \ln x &= 2.5 \end{aligned}$$

(2) Then set both sides of the equation as exponents of e,

$$e^{\ln x} = e^{2.5}$$

$$\Rightarrow x = e^{2.5}$$

(b) (1) First solve algebraically for $e^{f(x)}$,

$$\begin{aligned} 3e^{x-4} &= 24 \\ e^{x-4} &= \frac{24}{3} \\ e^{x-4} &= 8 \end{aligned}$$

(2) Then take natural logarithm of both sides,

$$\ln e^{x-4} = \ln 8$$

Solving we get,

$$x - 4 = \ln 8$$

$$\Rightarrow x = 4 + \ln 8$$

$\ln 8 = 2.07944$,

$$\Rightarrow x = 4 + 2.07944 = 6.07944$$

(c) (1) First solve algebraically for e^{3x},

$$2e^{3x} = 3616$$

$$e^{3x} = 1808$$

(2) Then take natural logarithm of both sides,

$$\ln e^{3x} = \ln 1808$$

Solving we get,

$$3x = \ln 1808$$

Using calculator or tables,

$$3x = 7.5$$

$$\Rightarrow x = 2.5$$

(d) (1) First solve algebraically for e^{x^2},

$$\frac{1}{3}e^{x^2} = 2701$$

$$e^{x^2} = 8103$$

(2) Then take natural logarithm of both sides,

$$\ln e^{x^2} = \ln 8103$$

Solving we get,

$$x^2 = \ln 8103$$

Using calculator or tables,

$$x^2 = 9$$

$$\Rightarrow x = \pm 3$$

■

6.6 Derivatives of Natural Exponential and Logarithmic Functions

The following are rules of differentiation

1. Given $f(x) = e^{g(x)}$ where $g(x)$ is a differentiable function of x, the derivative is

$$f'(x) = e^{g(x)} \cdot g'(x) \tag{6.1}$$

2. Given $f(x) = \ln[g(x)]$ where $g(x)$ is positive and differentiable function of x, the derivative is

$$f'(x) = \frac{1}{g(x)} \cdot g'(x) = \frac{g'(x)}{g(x)} \tag{6.2}$$

Problem 6.14. Find the derivatives of the following natural exponential functions:

(a) $f(x) = e^x$ (b) $f(x) = e^{3x}$ (c) $f(x) = e^{5x-2}$

(d) $f(x) = e^{4x^3}$ (e) $f(x) = 4e^{2-x^3}$

Solution.

(a) Let $g(x) = x$, then $g'(x) = 1$. Substituting in (6.1)

$$f'(x) = e^x \cdot 1 = e^x$$

(b) Let $g(x) = 3x$, then $g'(x) = 3$. Substituting in (6.1)

$$f'(x) = e^{3x} \cdot 3 = 3e^{3x}$$

(c) Let $g(x) = 5x - 2$, then $g'(x) = 5$. Substituting in (6.1)

$$f'(x) = e^{5x-2} \cdot 5 = 5e^{5x-2}$$

(d) Let $g(x) = 4x^3$, then $g'(x) = 12x^2$. Substituting in (6.1)

$$f'(x) = e^{4x^3} \cdot 12x^2 = 12x^2e^{4x^3}$$

(e) Let $g(x) = 2 - x^3$, then $g'(x) = -3x^2$. Substituting in (6.1)

$$f'(x) = 4e^{2-x^3} \cdot -3x^2 = -12x^2e^{2-x^3}$$

■

Problem 6.15. Combine rules to differentiate the following functions:

(a) $f(x) = 5xe^{3x}$ (b) $y = 6x^3e^{4x}$ (c) $y = (e^{3x} + e^{-5x})^4$

(d) $f(x) = \dfrac{e^{-6x}}{1 - 6x}$

Solution.

(a) $f(x) = 5xe^{3x}$. By product rule,

$$f'(x) = 5x(3e^{3x}) + e^{3x}(5) = 15xe^{3x} + 5e^{3x} = 5e^{3x}(3x + 1)$$

(b) $y = 6x^3e^{4x}$. By product rule,

$$y' = 6x^3(4e^{4x}) + e^{4x}(18x^2) = 24x^3e^{4x} + 18x^2e^{4x} = 6x^2e^{4x}(4x + 3)$$

(c) $y = (e^{3x} + e^{-5x})^4$. By the generalized power function rule or the chain rule,

$$y' = 4(e^{3x} + e^{-5x})^3 \cdot (3e^{3x} - 5e^{-5x}) = (12e^{3x} - 20e^{-5x})(e^{3x} + e^{-5x})^3$$

(d) $f(x) = \dfrac{e^{-6x}}{1-6x}$. By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{(1-6x)(-6e^{-6x}) - (e^{-6x})(-6)}{(1-6x)^2} \\ &= \frac{(-6e^{-6x} + 36xe^{-6x}) + 6e^{-6x}}{(1-6x)^2} \\ &= \frac{-6e^{-6x} + 36xe^{-6x} + 6e^{-6x}}{(1-6x)^2} \\ &= \frac{36xe^{-6x}}{(1-6x)^2} \end{aligned}$$

■

Problem 6.16. Find the derivatives of the following natural logarithmic functions:

(a) $f(x) = \ln x$ (b) $f(x) = \ln(-24x), x < 0$ (c) $f(x) = \ln 5x^4$

(d) $y = \ln(7x^2 - 15)$ (e) $y = \ln(x^2 + 9x + 12)$

Solution.

(a) Let $g(x) = x$, then $g'(x) = 1$. Substituting in (6.2)

$$f'(x) = \frac{g'(x)}{g(x)} = \frac{1}{x}$$

(b) Let $g(x) = -24x$, then $g'(x) = -24$. From (6.2)

$$f'(x) = \frac{g'(x)}{g(x)} = \frac{-24}{-24x} = \frac{1}{x}$$

(c) Let $g(x) = 5x^4$, then $g'(x) = 20x^3$. From (6.2)

$$f'(x) = \frac{g'(x)}{g(x)} = \frac{20x^3}{5x^4} = \frac{4}{x}$$

(d) Let $g(x) = 7x^2 - 15$, then $g'(x) = 14x$. Substituting in (6.2)

$$y' = \frac{g'(x)}{g(x)} = \frac{14x}{7x^2 - 15}$$

(e) Let $g(x) = x^2 + 9x + 12$, then $g'(x) = 2x + 9$. Substituting in (6.2)

$$y' = \frac{g'(x)}{g(x)} = \frac{2x + 9}{x^2 + 9x + 12}$$

■

Problem 6.17. Show that if $y = \ln kx$ then $y' = \dfrac{1}{x}$ and if $y = k \ln x$ then $y' = \dfrac{k}{x}$

Solution. $y = \ln kx$

Let $g(x) = kx$, then $g'(x) = k$. From (6.2)

$$y' = \frac{g'(x)}{g(x)} = \frac{k}{kx} = \frac{1}{x}$$

$y = k \ln x$, then

$$y' = k \cdot \frac{d}{dx}(\ln x)$$

Let $g(x) = x$, then $g'(x) = 1$. From (6.2)

$$\frac{d}{dx}(\ln x) = \frac{g'(x)}{g(x)} = \frac{1}{x}$$

$$\Rightarrow y' = k \cdot \frac{1}{x} = \frac{k}{x}$$

■

Problem 6.18. Combine rules to differentiate the following functions:

(a) $y = \ln^2 x = (\ln x)^2$ (b) $y = \ln^2 4x^3 = (\ln 4x^3)^2$ (c) $y = e^{2x} \ln 3x$

(d) $y = \ln(9x + 4)^2 \neq [\ln(9x + 4)]^2$ (e) $y = x^5 \ln x^3$ (f) $y = \dfrac{x}{\ln x}$

Solution.

(a) By the chain rule,

$$y' = 2(\ln x)\left(\frac{1}{x}\right) = \frac{2\ln x}{x}$$

(b) By the chain rule,

$$y' = 2(\ln 4x^3)\left(\frac{1}{4x^3}\right)(12x^2) = \frac{6\ln 4x^3}{x}$$

(c) By the product rule,

$$\begin{aligned} y' &= e^{2x}\left(\frac{1}{3x}\cdot 3\right) + (\ln 3x)(e^{2x}\cdot 2) \\ &= e^{2x}\left(\frac{1}{x}\right) + 2e^{2x}(\ln 3x) \\ &= e^{2x}\left(\frac{1}{x} + 2\ln 3x\right) \end{aligned}$$

(d) Let $g(x) = (9x+4)^2$, then $g'(x) = 2(9x+4)(9) = 18(9x+4)$. Substituting in (6.2),

$$y' = \frac{18(9x+4)}{(9x+4)^2} = \frac{18}{9x+4}$$

(e) By the product rule,

$$y' = x^5\left(\frac{1}{x^3}\right)(3x^2) + (\ln x^3)(5x^4) = 3x^4 + (\ln x^3)(5x^4)$$

(f) By the quotient rule,

$$y' = \frac{(\ln x)(1) - x(1/x)}{(\ln x)^2} = \frac{\ln x - 1}{\ln^2 x}$$

■

6.7 Logarithmic Differentiation

The natural logarithm function and its derivative can be used to facilitate the differentiation of product involving multiple terms. This process is called *logarithmic differentiation*. In this method we take natural logarithm of both sides before differentiation.

Problem 6.19. Find the derivative of

$$g(x) = (x^3 - 2)(x^2 - 3)(8x - 5) \tag{6.3}$$

using logarithmic differentiation.

Solution.

(1) Take the natural logarithm of both sides ,

$$\begin{aligned} \ln g(x) &= \ln[(x^3 - 2)(x^2 - 3)(8x - 5)] \\ &= \ln(x^3 - 2) + \ln(x^2 - 3) + \ln(8x - 5) \end{aligned}$$

(2) Take the derivative of $\ln g(x)$,

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)} = \frac{3x^2}{x^3 - 2} + \frac{2x}{x^2 - 3} + \frac{8}{8x - 5} \tag{6.4}$$

(3) Solve algebraically for $g'(x)$ in (6.4),

$$g'(x) = \left(\frac{3x^2}{x^3 - 2} + \frac{2x}{x^2 - 3} + \frac{8}{8x - 5}\right) \cdot g(x) \tag{6.5}$$

(4) Then substitute (6.3) for $g(x)$ in (6.5),

$$g'(x) = \left(\frac{3x^2}{x^3 - 2} + \frac{2x}{x^2 - 3} + \frac{8}{8x - 5}\right) [(x^3 - 2)(x^2 - 3)(8x - 5)]$$

■

Problem 6.20. Use logarithmic differentiation to find the derivative of

$$g(x) = \frac{(3x^5 - 4)(2x^3 + 9)}{7x^4 - 5} \tag{6.6}$$

Solution.

(1) Take the natural logarithm of both sides,

$$\begin{aligned}\ln g(x) &= \ln\left[\frac{(3x^5-4)(2x^3+9)}{7x^4-5}\right] \\ &= \ln(3x^5-4)+\ln(2x^3+9)-\ln(7x^4-5)\end{aligned}$$

(2) Take the derivative of $\ln g(x)$,

$$\frac{d}{dx}[\ln g(x)] = \frac{g'(x)}{g(x)} = \frac{15x^4}{3x^5-4}+\frac{6x^2}{2x^3+9}-\frac{28x^3}{7x^4-5} \tag{6.7}$$

(3) Solve algebraically for $g'(x)$ in (6.7),

$$g'(x) = \left(\frac{15x^4}{3x^5-4}+\frac{6x^2}{2x^3+9}-\frac{28x^3}{7x^4-5}\right)\cdot g(x) \tag{6.8}$$

(4) Then substitute (6.6) for $g(x)$ in (6.8),

$$g'(x) = \left(\frac{15x^4}{3x^5-4}+\frac{6x^2}{2x^3+9}-\frac{28x^3}{7x^4-5}\right)\cdot\frac{(3x^5-4)(2x^3+9)}{7x^4-5}$$

■

6.8 Practical Applications of Exponential Functions

Exponential functions are used in business, economics, and each of the different sciences to express rates of growth and decay, such as discrete and continuous interest compounding and discounting; the growth of human, insect, and animal populations; patterns and speed of learning and forgetting and rate of decay of elements and organisms.

We know that *compound interest* arises when interest is added to the *principal* of a deposit or loan, so that, from that moment on, the interest that has been added also earns interest. This addition of interest to the principal is called *compounding*. Interest Compounding is generally expressed in terms of exponential functions. A person lending out a principal P at interest rate r under terms of *annual compounding* for example, will have a value A at the end of one year equal to,

$$A_1 = P + rP = P(1 + r)$$

At the end of two years, he will have A_1 plus interest on A_1,

$$A_2 = A_1 + rA_1 = P(1 + r) + r[P(1 + r)] = P(1 + r)[1 + r] = P(1 + r)^2$$

At the end of t years, following the same procedure,

$$A_t = P(1 + r)^t \tag{6.9}$$

If the interest is compounded m times a year with the person receiving (r/m) interest m times during the course of a year, at the end of t years,

$$A_t = P\left(1 + \frac{r}{m}\right)^{mt} \tag{6.10}$$

If the interest is compounded *continuously* so that $m \to \infty$ then

$$A_t = Pe^{rt} \tag{6.11}$$

Problem 6.21. Find the value A of a principal P=$ 10,000 set out at an interest rate $r = 8\%$ for time $t = 3$ years when compounded (a) annually, (b) quarterly (c) monthly, and (d) continuously

Solution.

(a) Here $P = 10,000,\ r = 8\% = 0.08,\ t = 3$; substituting these values in (6.9) and using the y^x key of a calculator to estimate exponential functions throughout,

$$A = 10,000(1 + 0.08)^3 = 10,000(1.25971) = 12,597.10$$

(b) Substituting in (6.10) where $m = 4$ for quarterly compounding,

$$A = 10,000\left(1 + \frac{0.08}{4}\right)^{4(3)} = 10,000(1 + 0.02)^{12} = 12,682.40$$

(c) Now substituting $m = 12$ in (6.10) for monthly compounding,

$$A = 10,000\left(1 + \frac{0.08}{12}\right)^{12(3)} = 10,000(1 + 0.0067)^{36} = 12,702.40$$

(d) Finally using (6.11) for continuous compounding,

$$\begin{aligned} A &= 10,000e^{(0.08)3} = 10,000e^{0.24} \\ &= 10,000(1.27125) = 12,712.50 \end{aligned}$$

■

Problem 6.22. Find the value A of a principal P=\$ 100 set out at an interest rate $r = 8\%$ for time $t = 1$ year when compounded (a) annually, (b) semi-annually, (c) quarterly, and (d) continuously; (e) distinguish between the nominal and effective rates of interest. Use a calculator for exponential expressions.

Solution.

(a) Here $P = 100, r = 8\% = 0.08, t = 1$; substituting these values in (6.9) and using the y^x key of a calculator to estimate exponential functions throughout,

$$A = 100(1 + 0.08)^1 = 100(1.08) = 108$$

(b) Substituting in (6.10) with $m = 2$ for semi-annually compounding,

$$A = 100\left(1 + \frac{0.08}{2}\right)^{2(1)} = 100(1 + 0.04)^2 = 108.16$$

(c) Now substituting $m = 4$ in (6.10) for quarterly compounding,

$$A = 100\left(1 + \frac{0.08}{4}\right)^{4(1)} = 100(1 + 0.02)^4 = 108.24$$

(d) Finally using (6.11) for continuous compounding,

$$A = 100e^{(0.08)(1)} = 100e^{0.08} = 108.33$$

(e) In all four instances the stated or *nominal interest rate* is the same, namely 8%; the actual interest earned, however, varies according to the type of compounding. The *effective interest rate* on multiple compoundings is the comparable rate the bank would have to pay if interest were paid only once a year; that is 8.16% to equal semi-annual compounding, 8.24 % to equal quarterly compounding, and 8.33% to equal continuous compounding.

■

Problem 6.23. Find the value A of a principal P=$ 2,000 set out at an interest rate $r = 6\%$ for time $t = 5$ years when compounded (a) annually, (b) semi-annually, (c) quarterly, and (d) continuously.

Solution.

(a) Here $P = 2,000, r = 6\% = 0.06, t = 5$; substituting these values in (6.9) and using the y^x key of a calculator to estimate exponential functions throughout,

$$A = 2,000(1 + 0.06)^5 = 2,000(1.06)^5 = 2,676.45$$

(b) From (6.10) with $m = 2$,

$$A = 2,000\left(1 + \frac{0.06}{2}\right)^{2(5)} = 2,000(1 + 0.03)^{10} = 2,687.83$$

(c) From (6.10) with $m = 4$,

$$A = 2,000\left(1 + \frac{0.06}{4}\right)^{4(5)} = 2,000(1 + 0.015)^{20} = 2,693.71$$

(d) From (6.11),

$$A = 2,000e^{(0.06)(5)} = 2,000e^{0.3} = 2,699.72$$

■

Problem 6.24. How many years t will it take a sum of money P to double at 8% interest compounded annually?

Solution.

$$A = P(1 + 0.08)^t$$

For money to double, $A = 2P$. Substitute for A,

$$2P = P(1 + 0.08)^t$$

Divided by P,

$$2 = (1 + 0.08)^t$$

Take the natural log,

$$\begin{aligned} \ln 2 &= \ln(1 + 0.08)^t \\ \ln 2 &= t\ln(1 + 0.08) \\ \ln 2 &= t\ln 1.08 \\ 0.69315 &= 0.07696t \end{aligned}$$

Divide by 0.07696,

$$t = 9 \;\; years$$

■

Problem 6.25. How long it take money to treble at 10% interest compounded quarterly?

Solution. From (6.10),

$$A = P\left(1 + \frac{0.10}{4}\right)^{4(t)} = P(1 + 0.025)^{4t}$$

For money to treble, $A = 3P$. Substitute for A,

$$3P = P(1 + 0.025)^{4t}$$

$$3 = (1 + 0.025)^{4t}$$

Take the natural log,

$$\begin{aligned} \ln 3 &= \ln(1 + 0.025)^{4t} \\ \ln 3 &= 4t\ln(1 + 0.025) \end{aligned}$$

$$\begin{aligned} \ln 3 &= 4t \ln 1.025 \\ 1.09861 &= 4(0.02469)t \\ 1.09861 &= 0.09876t \end{aligned}$$

Divide by 0.09876,

$$t = 11.12 \ \ years$$

. ■

Discounting is the process of determining the present value P of a sum of money A to be received in the future.

Problem 6.26. Find the formula for discounting under (a) annual compounding, (b) multiple compounding and (c) continuous compounding.

Solution.

(a) Under annual compounding,

$$A = P(1+r)^t$$

Solve for P,

$$P = \frac{A}{(1+r)^t} = A(1+r)^{-t} \tag{6.12}$$

(b) Under multiple compounding,

$$A = P\left(1 + \frac{r}{m}\right)^{mt}$$

Solve for P,

$$P = \frac{A}{\left(1 + \frac{r}{m}\right)^{mt}} = A\left(1 + \frac{r}{m}\right)^{-mt} \tag{6.13}$$

(c) For continuous compounding,

$$A = Pe^{rt}$$

Solve for P,

$$P = Ae^{-rt} \tag{6.14}$$

■

Problem 6.27. Find the present value of \$1000 to be paid five years from now when the current interest rate is 10% if interest is compounded (a) annually, (b) quarterly, and (c) continuously.

Solution.

(a) From (6.12),

$$P = A(1+r)^{-t}$$

Substituting,

$$P = 1000(1+0.10)^{-5}$$

Using calculator,

$$P = 1000(0.62092) = 620.92$$

(b) From (6.13),

$$P = A\left(1+\frac{r}{m}\right)^{-mt}$$

Substituting,

$$P = 1000\left(1+\frac{0.10}{4}\right)^{-4(5)} = 1000(1+0.025)^{-20}$$

Using calculator,

$$P = 1000(0.61027) = 610.27$$

(c) From (6.14),

$$P = Ae^{-rt}$$

Substituting,

$$P = 1000e^{-(0.10)(5)} = 1000e^{-0.5}$$

$$= 1000(0.60653) = 606.53$$

■

Problem 6.28. Find the present value of \$1200 to be paid in 10 years from now when the current interest rate is 8% if interest is compounded (a) annually, (b) quarterly, and (c) continuously.

Solution.

(a) From (6.12),

$$P = A(1+r)^{-t}$$

Substituting,

$$P = 1200(1+0.08)^{-10}$$

Using calculator,

$$P = 1200(0.46319) = 555.83$$

(b) From (6.13),

$$P = A\left(1+\frac{r}{m}\right)^{-mt}$$

Substituting,

$$P = 1200\left(1+\frac{0.08}{4}\right)^{-4(10)} = 1200(1+0.02)^{-40}$$

Using calculator,

$$P = 1200(0.45289) = 543.47$$

(c) From (6.14),

$$P = Ae^{-rt}$$

Substituting,

$$P = 1200e^{-(0.08)(10)} = 1000e^{-0.8}$$

$$= 1200(0.44933) = 539.19$$

■

Problem 6.29. Find the percentage or relative rate of growth G for each of the following functions using the derivative of the natural logarithm of the function:

(a) $f(t) = 5t^2$ (b) $f(t) = \dfrac{t}{4^t}$

Solution.

(a) First take the natural logarithm of both sides,

$$\ln f(t) = \ln 5t^2$$

$$\ln f(t) = \ln 5 + 2 \ln t$$

Then take derivative, note that $\ln 5$ is a constant,

$$G = \frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)} = 0 + 2\left(\frac{1}{t}\right) = \frac{2}{t}$$

(b) First take the natural logarithm of both sides,

$$\ln f(t) = \ln \frac{t}{4^t}$$

$$\ln f(t) = \ln t - t \ln 4$$

Then take derivative, note that $\ln 4$ is a constant,

$$G = \frac{d}{dt} \ln f(t) = \frac{f'(t)}{f(t)} = \frac{1}{t} - \ln 4$$

■

Problem 6.30. Find the relative growth rate G of sales at $t = 4$, given $S(t) = 100000e^{0.5\sqrt{t}}$.

Solution. First take the natural logarithm of both sides,

$$\ln S(t) = \ln 100000 + 0.5\sqrt{t}$$

$$\ln S(t) = \ln 100000 + 0.5t^{1/2}$$

Then take derivative,

$$G = \frac{d}{dt}\ln S(t) = \frac{S'(t)}{S(t)} = 0.5\left(\frac{1}{2}\right)t^{-1/2} = \frac{0.25}{\sqrt{t}}$$

At $t = 4$,

$$G = \frac{0.25}{\sqrt{4}} = 0.125 = 12.5\%$$

■

Problem 6.31. Derive the formula for finding the effective rate of interest r_e for multiple compondings when $t > 1$.

Solution. From the explanation of the effective rate of interest in Problem 6.22 (e), we can write

$$P(1 + r_e)^t = P\left(1 + \frac{r}{m}\right)^{mt}$$

Dividing by P and taking t^{th} root of each side,

$$\begin{aligned} 1 + r_e &= \left(1 + \frac{r}{m}\right)^m \\ r_e &= \left(1 + \frac{r}{m}\right)^m - 1 \end{aligned} \quad (6.15)$$

For continuous compounding,

$$\begin{aligned} P(1 + r_e)^t &= Pe^{rt} \\ 1 + r_e &= e^r \\ r_e &= e^r - 1 \end{aligned} \quad (6.16)$$

■

Problem 6.32. Find the effective rate of interest for $P = \$2000$ ar $r = 6\%$ when compounded (a) semiannually, (b) quarterly, (c) continuously, noting that P and t do not matter.

Solution.

(a) From (6.15),

$$r_e = \left(1 + \frac{r}{m}\right)^m - 1$$

Substituting,

$$\begin{aligned} r_e &= \left(1+\frac{.06}{2}\right)^2 - 1 \\ &= (1+0.03)^2 - 1 \\ &= 1.0609 - 1 = 0.0609 = 6.09\% \end{aligned}$$

(b)

$$\begin{aligned} r_e &= \left(1+\frac{.06}{4}\right)^4 - 1 \\ &= (1+0.015)^4 - 1 \\ &= 1.06136 - 1 = 0.06136 = 6.136\% \end{aligned}$$

(c) From (6.16),

$$r_e = e^r - 1$$

Substituting,

$$r_e = e^{0.06} - 1 = 1.061837 - 1 = 0.061837 = 6.1837\%$$

■

6.9 Exercises

1. Solve the following for x, y, a by finding the equivalent expression:

   (a) $y = \log_2 \frac{1}{16}$

   (b) $\log_a 4 = \frac{2}{3}$

   (c) $\log_{81} x = \frac{3}{4}$

2. Use the properties of logarithms to write the following as sums, differences, or products:

   (a) $\log_a u^2 v^{-3}$

   (b) $\log_a \sqrt[3]{x}$

3. Use the properties of logarithms to write the following natural logarithmic forms as sums, differences, or products:

   (a) $\ln \frac{x^3}{y^4}$

   (b) $\ln \frac{4x}{7y}$

   (c) $\ln(x^3\sqrt{y})$

4. Differentiate the following functions:

   (a) $y = (e^{-2x})^3$

   (b) $y = \frac{e^{7x} - 1}{e^{7x} + 1}$

   (c) $y = \ln(9x^2 + 5)$

   (d) $y = 15 \ln x$

   (e) $y = e^{2x} \ln 3x$

   (f) $y = \ln^2(21x + 8)$

5. Use logarithmic differentiation to find the derivative of

$$g(x) = (x^4 + 7)(x^5 + 6)(x^3 + 2)$$

6. Find the effective rate of interest for $P = \$500$ ar $r = 12\%$ when compounded (a) semiannually, (b) quarterly, (c) continuously, noting that P and t do not matter.

7. Find the relative growth of profits at $t = 8$, given $\pi(t) = 2, 50, 000e^{1.2t^{1/3}}$

Chapter 7

Integration

7.1 Antidifferentiation

Integration is the inverse operation of differentiation.Reversing the process of differentiation and finding the original function from the derivative is called *integration* or *antidifferentiation*. Suppose the derivative of $F(x)$ is $F'(x)$.Then $F(x)$ is called the integral or antiderivative of $F'(x)$.

Let $f(x) = F'(x)$, the antiderivative of $f(x)$ is expressed as

$$\int f(x)dx = F(x) + c$$

Here the left-hand side of the equation is read as "the *indefinite integral* of f of x with respect to x ". The symbol $\int$ is an *integral sign*, $f(x)$ is the *integrand*, and c is the *constant of integration*. The integral is called an indefinite integral because its value is not fully determined until the end points are specified. This ambiguity is dealt with by the addition of the constant c at the end.

7.2 Rules for Indefinite Integrals

The following rules for indefinite integrals are obtained by reversing the corresponding rules of differentiation.

RULE 1. The integral of a constant k is

$$\int kdx = kx + c \tag{7.1}$$

RULE 2. The integral of 1, written simply as dx is

$$\int dx = x + c \tag{7.2}$$

RULE 3. The integral of a power function x^n, where $n \neq -1$, is

$$\int x^n dx = \frac{x^{n+1}}{n+1} + c \qquad n \neq -1 \tag{7.3}$$

RULE 4. The integral of x^{-1}, or $(1/x)$, is

$$\int x^{-1} dx = \ln|x| + c \qquad x \neq 0 \tag{7.4}$$

Note that if $x > 0$,

$$\int x^{-1} dx = \ln x + c$$

RULE 5. The integral of a natural exponential function is

$$\int e^{kx} dx = \frac{e^{kx}}{k} + c \tag{7.5}$$

RULE 6. The integral of a constant times a function equals the constant times the antiderivative of the function.

$$\int kf(x)dx = k \int f(x)dx + c \tag{7.6}$$

RULE 7. The integral of the sum or difference of two or more functions equals the sum or difference of their integrals.

$$\int [f(x) \pm g(x)]dx = \int f(x)dx \pm \int g(x)dx \tag{7.7}$$

RULE 8. The integral of the negative of a function equals the negative of the integral of the function

$$\int -f(x)dx = -\int f(x)dx \tag{7.8}$$

Example 7.1. The rules for indefinite integrals are illustrated below.

(a) $\int 5dx = 5x + c$ [Rule 1]

(b) $\int x^4 dx = \dfrac{x^{4+1}}{4+1} + c = \dfrac{x^5}{5} + c$ [Rule 3]

(c) $\int 5x^4 dx = 5 \int x^4 dx$ [Rule 6]

. $= 5\left(\dfrac{x^{4+1}}{4+1} + c_1\right)$ [Rule 3]

. $= 5\left(\dfrac{x^5}{5} + c_1\right) = x^5 + 5c_1$

. $= x^5 + c$

where c_1 and c are arbitrary constants and $5c_1 = c$.

(d) $\int x^{-3} dx = \dfrac{x^{-3+1}}{-3+1} + c = -\dfrac{x^{-2}}{2} + c$ [Rule 3]

(e) $\int e^{5x} dx = \dfrac{e^{5x}}{5} + c$ [Rule 5]

(f) $\int (1 - x) dx = \int dx - \int x dx$ [Rule 8]

. $= x - \dfrac{x^2}{2} + c$ [Rule 3]

Problem 7.1. Find the following indefinite integrals:

(a) $\displaystyle\int 7dx$

(b) $\displaystyle\int -12dx$

(c) $\displaystyle\int x^5 dx$

(d) $\displaystyle\int \frac{4}{x^5} dx$

(e) $\displaystyle\int (20x^4 - 8x^3) dx$

(f) $\displaystyle\int \sqrt{x} dx$

(g) $\int 12e^{-3t}dt$

(h) $\int \frac{2}{x}dx$

(i) $\int \frac{1}{x+9}dx$

(j) $\int \sqrt{x+5}dx$

Solution.

(a) $\int 7dx = 7x + c$

(b) $\int -12dx = -\int 12dx = -12x + c$

(c) $\int x^5 dx = \frac{x^6}{6} + c$

(d)

$$\begin{aligned}\int \frac{4}{x^5}dx &= 4\int \frac{1}{x^5}dx \\ &= 4\int x^{-5}dx \\ &= 4\left(\frac{x^{-5+1}}{-5+1} + c_1\right) \\ &= 4\left(\frac{x^{-4}}{-4} + c_1\right) = -x^{-4} + 4c_1 \\ &= -\frac{1}{x^4} + c \qquad \textit{where} \;\; c = 4c_1\end{aligned}$$

(e)

$$\begin{aligned}\int (20x^4 - 8x^3)dx &= \int 20x^4 dx - \int 8x^3 dx \\ &= 20\int x^4 dx - 8\int x^3 dx \\ &= 20 \cdot \frac{x^5}{5} - 8 \cdot \frac{x^4}{4} + c \\ &= \frac{4x^5}{5} - 2x^4 + c\end{aligned}$$

(f)

$$\begin{aligned}\int \sqrt{x}dx &= \int x^{1/2}dx \\ &= \frac{x^{1/2+1}}{1/2+1}+c \\ &= \frac{x^{3/2}}{3/2}+c \\ &= \frac{2}{3}x^{3/2}+c\end{aligned}$$

(g) $\int 12e^{-3t}dt = 12\int e^{-3t}dt = 12\left(\frac{e^{-3t}}{-3}+c_1\right) = -4e^{-3t}+12c_1 = -4e^{-3t}+c$

(h) $\int \frac{2}{x}dx = 2\int \frac{1}{x}dx = 2\ln|x|+c = \ln x^2 + c$

(i) $\int \frac{1}{x+9}dx = \int (x+9)^{-1}dx = \ln|x+9|+c$

(j)

$$\begin{aligned}\int \sqrt{x+5}dx &= \int (x+5)^{1/2}dx \\ &= \frac{(x+5)^{1/2+1}}{1/2+1}+c \\ &= \frac{(x+5)^{3/2}}{3/2}+c \\ &= \frac{2}{3}(x+5)^{3/2}+c\end{aligned}$$

■

Problem 7.2. Find the antiderivative for each of the following :

(a) $\int (25x^{1/4}+16x^{1/3})dx$, given $F(0) = 19$

(b) $\int (16e^{2t}+15e^{-3t})dt$, given $F(0) = 9$

(c) $\int (4x^{-1}+5x^{-2})dx$, given $F(1) = 3$

Solution.

(a)

$$\begin{aligned}
\int (25x^{1/4} + 16x^{1/3})dx &= \int 25x^{1/4}dx + \int 16x^{1/3}dx \\
&= 25\int x^{1/4}dx + 16\int x^{1/3}dx \\
&= 25\left(\frac{x^{1/4+1}}{1/4+1}\right) + 16\left(\frac{x^{1/3+1}}{1/3+1}\right) + c \\
&= 25\left(\frac{x^{5/4}}{\frac{5}{4}}\right) + 16\left(\frac{x^{4/3}}{\frac{4}{3}}\right) + c \\
&= 25\left(\frac{4}{5}\right)x^{5/4} + 16\left(\frac{3}{4}\right)x^{4/3} + c \\
&= 20x^{5/4} + 12x^{4/3} + c \\
\Rightarrow F(x) &= 20x^{5/4} + 12x^{4/3} + c
\end{aligned}$$

Given $F(0) = 19, \ \Rightarrow 19 = 20(0)^{5/4} + 12(0)^{4/3} + c = 0 + c = c \ \Rightarrow c = 19$

$$\therefore \qquad F(x) = 20x^{5/4} + 12x^{4/3} + 19$$

(b)

$$\begin{aligned}
\int (16e^{2t} + 15e^{-3t})dt &= \int 16e^{2t}dt + \int 15e^{-3t}dt \\
&= 16\int e^{2t}dt + 15\int e^{-3t}dt \\
&= 16\left(\frac{e^{2t}}{2}\right) + 15\left(\frac{e^{-3t}}{-3}\right) + c \\
&= 8e^{2t} - 5e^{-3t} + c \\
\Rightarrow F(t) &= 8e^{2t} - 5e^{-3t} + c
\end{aligned}$$

Given $F(0) = 9$, $\Rightarrow 9 = 8e^{2(0)} - 5e^{-3(0)} + c \Rightarrow 9 = 8 - 5 + c = 3 + c$
$\Rightarrow c = 9 - 3 = 6$

$$\therefore \qquad F(t) = 8e^{2t} - 5e^{-3t} + 6$$

(c) Try your self . Answer is

$$F(x) = 4\ln x - \frac{5}{x} + 8$$

■

7.3 The Definite Integral

The area under the curve $y = f(x)$ between $x = a$ and $x = b$ is the *definite integral* of $f(x)$ over the interval a to b. We write,

$$Area \;\; A = \int_a^b f(x)dx$$

We read right-hand side as "the integral from a to b of f of x dx "; a is called the *lower limit* of integration, b the *upper limit* of integration.

7.4 The Fundamental Theorem of Calculus

If $f(x) = F'(x)$ then

$$\int_a^b f(x)dx = F(x)|_a^b = F(b) - F(a) \tag{7.9}$$

where the symbol $|_a^b$ or $]_a^b$ or $[\cdots]_a^b$ indicates that b and a are to be substituted successively for x.

Problem 7.3. Evaluate the following definite integrals:

(a) $\int_1^5 4xdx$

(b) $\int_2^4 (9x^2 + 6)dx$

Solution.

(a) $$\int_1^5 4x dx = 2x^2\Big|_1^5 = 2(5)^2 - 2(1) = 50 - 2 = 48$$

(b) $$\int_2^4 (9x^2 + 6)dx = [3x^3 + 6x]_2^4 = [3(4)^3 + 6(4)] - [3(2)^3 + 6(2)] = 180$$

■

Problem 7.4. Evaluate the following definite integrals:

(a) $\int_1^9 12\sqrt{x}dx$

(b) $\int_0^1 e^{t/2}dt$

(c) $\int_0^2 (7 + x)^3 dx$

Solution.

(a) $$\int_1^9 12\sqrt{x}dx = \int_1^9 12x^{1/2}dx = 8x^{3/2}\Big|_1^9 = 8(9)^{3/2} - 8(1)^{3/2} = 8(27) - 8 = 208$$

(b) $$\int_0^1 e^{t/2}dt = \left[\frac{e^{t/2}}{1/2}\right]_0^1 = \left[2e^{t/2}\right]_0^1 = 2e^{1/2} - 2e^{0/2} = 2e^{1/2} - 2 = 2(e^{1/2} - 1)$$

(c) $$\int_0^2 (7 + x)^3 dx = \left[\frac{(7 + x)^4}{4}\right]_0^2 = \frac{(7 + 2)^4}{4} - \frac{(7 + 0)^4}{4} = \frac{9^4 - 7^4}{4} = 1040$$

■

Problem 7.5. Determine the area under the curve $y = 20 - 4x$ over the interval 0 to 5.

Solution.

$$Area\ A = \int_0^5 (20 - 4x)dx = [20x - 2x^2]_0^5 = 20(5) - 2(5)^2 = 100 - 50 = 50$$

■

7.5 Properties of Definite Integrals and Area Between Curves

1. Reversing the order of the limits of integration changes the sign of the definite integral.

$$\int_a^b f(x)dx = -\int_b^a f(x)dx \tag{7.10}$$

2. If the upper limit of integration equals the lower limit of integration, the value of the definite integral is zero.

$$\int_a^a f(x)dx = F(a) - F(a) = 0 \tag{7.11}$$

3. The definite integral can be expressed as the sum of component sub integrals.

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx \qquad a \le b \le c \tag{7.12}$$

4.

$$\int_a^b f(x)dx \pm \int_a^b g(x)dx = \int_a^b [f(x) \pm g(x)]dx \tag{7.13}$$

5.

$$\int_a^b kf(x)dx = k\int_a^b f(x)dx \tag{7.14}$$

Problem 7.6. Draw graphs for the following functions and evaluate the area between the curves over the stated interval

(a) $y = 7 - x^2$ and $y = 3$ from $x = -2$ to $x = 2$.

(b) $y = 10$ and $y = x^2 + 1$ from $x = -3$ to $x = 3$.

Solution.

(a) The desired area is the area under the curve specified by $y = 7 - x^2$ from $x = -2$ to $x = 2$, minus the area under the curve specified by $y = 3$ from $x = -2$ to $x = 2$.

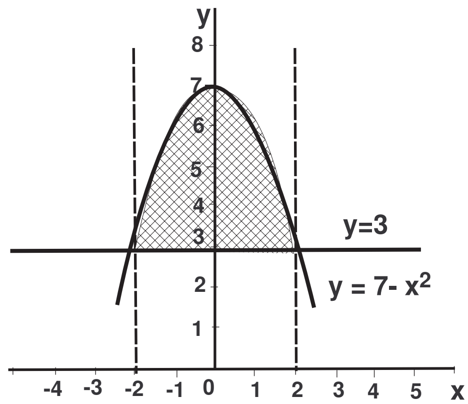

Figure 7.1

$$\begin{aligned}
A &= \int_{-2}^{2}(7-x^2)dx - \int_{-2}^{2}3dx \\
&= \int_{-2}^{2}[(7-x^2)-(3)]dx \\
&= \int_{-2}^{2}(7-x^2-3)dx \\
&= \int_{-2}^{2}(4-x^2)dx \\
&= \left[4x-\frac{x^3}{3}\right]_{-2}^{2} \\
&= \left(4(2)-\frac{(2)^3}{3}\right)-\left(4(-2)-\frac{(-2)^3}{3}\right) \\
&= \left(8-\frac{8}{3}\right)-\left(-8-\frac{-8}{3}\right) \\
&= \left(8-\frac{8}{3}\right)-\left(-8+\frac{8}{3}\right) \\
&= \left(\frac{16}{3}\right)-\left(\frac{-16}{3}\right) \\
&= \left(\frac{16}{3}+\frac{16}{3}\right)=\frac{32}{3}
\end{aligned}$$

So the area between the curves is $A = \frac{32}{3}$.

(b) .

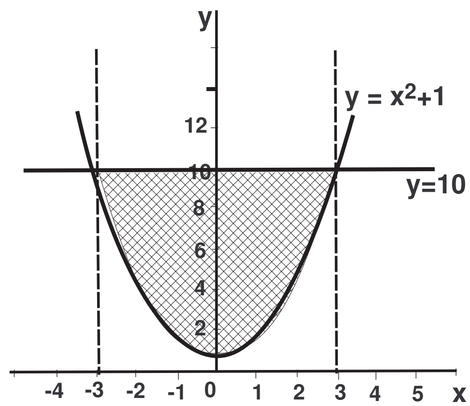

Figure 7.2

Required area

$$
\begin{aligned}
A &= \int_{-3}^{3} 10dx - \int_{-3}^{3} (x^2+1)dx \\
&= \int_{-3}^{3} [10-(x^2+1)]dx = \int_{-3}^{3} (9-x^2)dx \\
&= \left[9x - \frac{x^3}{3}\right]_{-3}^{3} \\
&= \left[9(3) - \frac{(3)^3}{3}\right] - \left[9(-3) - \frac{(-3)^3}{3}\right] = 36
\end{aligned}
$$

So the area between the curves is $A = 36$.

■

7.6 Average Value of a Function and The Volume of a Solid of Revolution

If $f(x)$ is a continuous function on the interval $[a, b]$, the *average value* m of $f(x)$ on this interval is defined as

$$m = \frac{1}{b-a}\int_a^b f(x)dx \tag{7.15}$$

If a continuous function $f(x)$ from $x = a$ to $x = b$ is rotated around the x-axis, a solid of revolution is created. The *volume of the solid of revolution* V can be expressed mathematically as

$$V = \int_a^b \pi[f(x)]^2 dx \tag{7.16}$$

Problem 7.7. Find the average value m of the following functions on the interval $[a, b]$:

(a) $f(x) = x^2 - 1;\ a = 0,\ b = 3$

(b) $f(x) = \dfrac{1}{\sqrt{x+3}};\ a = 1,\ b = 6$

(c) $f(x) = e^{x/6};\ a = 0,\ b = 6$

Solution.

(a) From (7.15), $\quad m = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)dx$

Substituting we get,

$$\begin{aligned} m &= \frac{1}{3-0}\int_0^3 (x^2-1)dx \\ &= \frac{1}{3}\left[\frac{x^3}{3} - x\right]_0^3 \\ &= \frac{1}{3}\left[\left(\frac{(3)^3}{3} - 3\right) - \left(\frac{(0)^3}{3} - 0\right)\right] = \frac{1}{3}(9-3) = \frac{6}{3} = 2 \end{aligned}$$

Average value $m = 2$.

(b) From (7.15), $m = \frac{1}{b-a}\int_a^b f(x)dx$

Substituting we get,

$$\begin{aligned} m &= \frac{1}{6-1}\int_1^6 \frac{1}{\sqrt{x+3}}dx \\ &= \frac{1}{5}\int_1^6 (x+3)^{-1/2}dx \\ &= \frac{1}{5}\left[\frac{(x+3)^{1/2}}{1/2}\right]_1^6 \\ &= \frac{1}{5}[2(x+3)^{1/2}]_1^6 = \frac{2}{5}[(x+3)^{1/2}]_1^6 \\ &= \frac{2}{5}[9^{1/2} - 4^{1/2}] = \frac{2}{5}(3-2) = \frac{2}{5} \end{aligned}$$

Average value $m = \frac{2}{5}$.

(c) From (7.15), $m = \frac{1}{b-a}\int_a^b f(x)dx$

$$\begin{aligned} m &= \frac{1}{6-0}\int_0^6 e^{x/6}dx \\ &= \frac{1}{6}\left[\frac{e^{x/6}}{1/6}\right]_0^6 \\ &= \frac{1}{6}\left[6e^{x/6}\right]_0^6 \\ &= \left[e^{x/6}\right]_0^6 = e^{(1)} - e^{(0)} \\ &= e - 1 = 1.71828 \end{aligned}$$

Average value $m = 1.71828$.

■

Problem 7.8. Find the volume V of the solid of revolution generated by revolving around the x-axis the regions under each of the following curves.

(a) $f(x) = 5x^2$; $a = 1, b = 3$

(b) $f(x) = \sqrt{9 - x^2}$; $a = -3, b = 3$

Solution.

(a) From (7.16),

$$V = \int_a^b \pi[f(x)]^2 dx$$

Substituting,

$$V = \int_1^3 \pi[(5x^2)]^2 dx$$

Squaring and rearranging constants,

$$V = \int_1^3 \pi(25x^4)dx = 25\pi \int_1^3 x^4 dx = 25\pi \left[\frac{1}{5}x^5\right]_1^3 = 5\pi(243 - 1) = 1210\pi.$$

(b) From (7.16),

$$V = \int_a^b \pi[f(x)]^2 dx$$

Substituting,

$$V = \int_{-3}^3 \pi[\sqrt{9 - x^2}]^2 dx$$

Squaring and rearranging constants,

$$V = \int_{-3}^3 \pi(9 - x^2)dx = \pi \left[9x - \frac{x^3}{3}\right]_{-3}^3 = \pi[(27 - 9) - (-27 + 9)] = 36\pi.$$

■

7.7 Practical Applications

Problem 7.9. A person's rate of reaction or sensitivity to a specific drug t hours after it is administered is given by

$$S'(t) = \frac{3}{t} + \frac{4}{t^2}$$

where S is measured is suitable units. Find the strength of the total reaction from $t = 1$ to $t = 8$.

Solution.

$$S(t) = \int_1^8 (3t^{-1} + 4t^{-2})dt = \left(3\ln t - \frac{4}{t}\right)\Big|_1^8 = \left(3\ln 8 - \frac{1}{2}\right) - \left(3\ln 1 - \frac{4}{1}\right) \approx 9.7$$

■

Problem 7.10. A firm's marginal cost function is $C'(x) = x^2 - 4x + 110$ with x representing the number of units per day. Fixed costs are 340 a day. What is the total cost $C(x)$ of producing x units per day?

Solution.

$$C(x) = \int (x^2 - 4x + 110)dx = \frac{x^3}{3} - 2x^2 + 110x + c$$

Substituting $C(0) = 340$,

$$340 = \frac{(0)^3}{3} - 2(0)^2 + 110(0) + c \qquad \Rightarrow c = 340$$

and

$$C(x) = \frac{x^3}{3} - 2x^2 + 110x + 340$$

■

Problem 7.11. A manufacturer's marginal profit is $\pi' = -3x^2 + 80x + 140$. Find the profit π earned by increasing production from two units to four units.

Solution.

$$\begin{aligned} \pi(4) - \pi(2) &= \int_2^4 (-3x^2 + 80x + 140)dx \\ . \qquad &= (-x^3 + 40x^2 + 140x)\Big|_2^4 = 704 \end{aligned}$$

■

Problem 7.12. A zinc mine extracts ore at a rate of thousands of tons per year, $Z'(t) = 18t - (22/\sqrt{t})$. Find the total amount of ore extracted from year 0 to year 9

Solution.

$$Z(t) = \int_0^9 (18t - 22t^{-1/2})dt = (9t^2 - 44t^{1/2})\Big|_0^9 = 5,97,000 \; tons$$

■

Problem 7.13. According to engineer's estimates, the cost of a new product is $C = 6\sqrt{x} + 15$. Find the average cost m of producing the first 64 units.

Solution. From (7.15),

$$m = \frac{1}{b-a}\int_a^b f(x)dx = \frac{1}{64-0}\int_0^{64} 6\sqrt{x} + 15dx$$

$$. = \frac{1}{64}\left[4x^{3/2} + 15x\right]_0^{64} = 47$$

■

7.8 Exercises

1. Determine the following indefinite integrals:
   (a) $\displaystyle\int 10e^{t/2}dt$
   (b) $\displaystyle\int \frac{dx}{\sqrt{x}}dx$
   (c) $\displaystyle\int \frac{8}{t^5}dt$
   (d) $\displaystyle\int 6(x-15)^{-2}dx$
2. Evaluate the following definite integrals:
   (a) $\displaystyle\int_2^4 3x^2dx$
   (b) $\displaystyle\int_4^{36} \frac{dx}{\sqrt{x}}dx$

(c) $\int_0^1 12e^{-4t}dt$

(d) $\int_1^5 3x^{-1}$

3. Draw graphs for the following functions and evaluate the area between the curves over the stated interval.

   (a) $y = 7 - x$ and $y = 4x - x^2$ from $x = 1$ to $x = 4$

   (b) $y = 6 - x$ and $y = 4$ from $x = 0$ to $x = 5$

   (c) $y = 10 - 2x$ and $y = x + 1$ from $x = 0$ to $x = 4$

   (d) The area bounded by $y = x^2$,$y = 6x$, and $y = 8x - 2$ from $x = 0$ to $x = 2$

4. Find the average value m of the following functions on the interval $[a, b]$:

   (a) $f(x) = 2x + 8;\ a = 3,\ b = 7$

   (b) $f(x) = \sqrt{x-2};\ a = 6,\ b = 11$

5. Find the volume V of the solid of revolution generated by revolving around the x-axis the regions under each of the following curves:

   (a) $f(x) = 3x + 2\ ;\ a = 0, b = 4$

   (b) $f(x) = e^{-1.5x}\ ;\ a = 0, b = 1$

6. A pipe on an offshore drilling platform is damaged ,spilling oil at a rate of $(35t+80)$ barrels per hour t. How many barrels will be leaked the first day?

7. A producer's marginal cost is $C'(x) = \frac{1}{12}x^2 - x + 180$. What is the total cost $C(x)$ of producing five extra units if three units are currently being produced?

8. Find the producer's surplus for the supply curve $p = x^2 + 4x + 60$ at the level $x_0 = 5$, $p_0 = 85$.

9. The population in millions of people for a newly emerging nation is $P(t) = 18e^{0.032t}$. Find the average population m over the next 25 years.

Chapter 8

Multivariable Calculus

8.1 Functions of Several Variables

In this section, we extend the definition of a function of one variable to functions of two or more variables. You will recall that a function was a rule which assigned a unique value to each input value. It is going to be similar for two or more variables. The only difference is that the input is not a value any more, it is several values. $z = f(x, y)$ is defined as a function of *two independent variables* if there exists one and only one value of z in the range of f for each ordered pair of real numbers (x, y) in the domain of f. Here z is the *dependent variable*, x and y are the *independent variables*.

Example 8.1. If a firm produces one good x for which the cost function is $C(x) = 350+8x$ and another good y for which the cost function is $C(y) = 225+6y$, the total cost to the firm can be expressed simply as

$$C(x, y) = 350 + 8x + 225 + 6y = 575 + 8x + 6y$$

Other examples of multi variable function include:

$$f(x, y) = x^3 + 3xy + y^2$$

$$z(x, y) = 5xy^2$$

Example 8.2. Multivariable functions can be evaluated for specific values of x and y such as $x = 2$ and $y = 4$ by replacing x and y with the desired values. Using the function from Example 8.1

$$C(2,4) = 575 + 8(2) + 6(4) = 615$$

$$f(2,4) = 2^3 + 3(2)(4) + 4^2 = 48$$

$$z(2,4) = 5(2)(4)^2 = 160$$

8.2 Partial Derivatives

Let $z = f(x, y)$. The *partial derivative of z with respect to x* measures the rate of change of z with respect to x while y is held constant. It is written $\partial z/\partial x, \partial f/\partial x, f_x(x,y), f_x$, or z_x and is defined as

$$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x} \tag{8.1}$$

Similarly, the *partial derivative of z with respect to y* measures the rate of change of z with respect to y while x is held constant. It is written $\partial z/\partial y, \partial f/\partial y, f_y(x,y), f_y$, or z_y and is defined as

$$\frac{\partial z}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y} \tag{8.2}$$

To find the partial derivative of a function with respect to one of the independent variables, simply treat the other independent variable as a constant and follow the ordinary rules of differentiation.

Problem 8.1. Find $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ where $z = 5x^3y^4$

Solution.

(a) When differentiating with respect to x, treat the y term as a constant by mentally joining it with the coefficient:

$$z = (5y^4) \cdot x^3$$

then take the derivative of the x term, holding the y term constant,

$$\frac{\partial z}{\partial x} = (5y^4) \cdot \frac{d}{dx}(x^3) = (5y^4) \cdot 3x^2 = 15x^2y^4$$

$$\therefore \quad \frac{\partial z}{\partial x} = 15x^2y^4$$

(b) When differentiating with respect to y, treat the x term as a constant by mentally joining it with the coefficient :

$$z = (5x^3) \cdot y^4$$

then take the derivative of the y term, holding the x term constant,

$$\frac{\partial z}{\partial y} = (5x^3) \cdot \frac{d}{dy}(y^4) = (5x^3) \cdot 4y^3 = 20x^3y^3$$

$$\therefore \quad \frac{\partial z}{\partial y} = 20x^3y^3$$

■

Problem 8.2. Find $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$ where $z = 6x^3 - 7x^2y^2 + 4y^5$

Solution.

(a) When differentiating with respect to x, mentally bracket off all y terms to remember to treat them as constants:

$$z = 6x^3 - (7y^2)x^2 + (4y^5)$$

then take the derivative of each term, recalling that while multiplicative constants remain in the differentiation process, additive constants drop out because the derivative of a constant is zero.

$$\begin{aligned} \frac{\partial z}{\partial x} &= \frac{d}{dx}(6x^3) - (7y^2) \cdot \frac{d}{dx}(x^2) + \frac{d}{dx}(4y^4) \\ &= 18x^2 - (7y^2) \cdot 2x + 0 \\ &= 18x^2 - 14xy^2 \end{aligned}$$

(b) When differentiating with respect to y, mentally bracket off all x terms to remember to treat them as constants:

$$z = (6x^3) - (7x^2)y^2 + 4y^5$$

then take the derivative of each term

$$\begin{aligned}\frac{\partial z}{\partial y} &= \frac{d}{dy}(6x^3) - (7x^2)\cdot\frac{d}{dy}(y^2) + \frac{d}{dy}(4y^5)\\ &= 0 - (7x^2)\cdot 2y + 20y^4\\ &= 14x^2y + 20y^4\end{aligned}$$

■

Problem 8.3. Find the first order partial derivatives for each of the following functions:

(a) $f(x, y) = 11x^4y^7$

(b) $f(x, y) = 4x^3 - 8xy - 7y^4$

(c) $f(x, y, z) = 8x^2y^4z^5$

Solution.

(a)

$$f_x(x, y) = (11y^7)\cdot\frac{d}{dx}(x^4) = (11y^7)\cdot(4x^3) = 44x^3y^7$$

$$f_y(x, y) = (11x^4)\cdot\frac{d}{dy}(y^7) = (11x^4)\cdot(7y^6) = 77x^4y^6$$

(b)

$$f_x(x, y) = 12x^2 - 8y$$

$$f_y(x, y) = -8x - 28y^3$$

(c)

$$\frac{\partial f}{\partial x} = 16xy^4z^5 \qquad \frac{\partial f}{\partial y} = 32x^2y^3z^5 \qquad \frac{\partial f}{\partial z} = 40x^2y^4z^4$$

■

8.3 Rules of Partial Differentiation

Partial differentiation follow the same basic patterns as the rules of differentiation.

Product Rule

Given $z = g(x, y) \cdot h(x, y)$,

$$\frac{\partial z}{\partial x} = g(x, y) \cdot \frac{\partial h}{\partial x} + h(x, y) \cdot \frac{\partial g}{\partial x} \tag{8.3}$$

$$\frac{\partial z}{\partial y} = g(x, y) \cdot \frac{\partial h}{\partial y} + h(x, y) \cdot \frac{\partial g}{\partial y} \tag{8.4}$$

Problem 8.4. Use the product rule to find the first-order partial derivatives for each of the following functions:

(a) $z = (7x + 2y)(5x + 9)$

(b) $f(x, y) = 7x^3(4x + 9y^2)$

Solution.

(a)

$$\frac{\partial z}{\partial x} = (7x + 2y)(5) + (5x + 9)(7) = 35x + 10y + 35x + 63 = 70x + 10y + 63$$

$$\frac{\partial z}{\partial y} = (7x + 2y)(0) + (5x + 9)(2) = 10x + 18$$

(b)

$$\begin{aligned} \frac{\partial f}{\partial x} &= 7x^3(4) + (4x + 9y^2)(21x^2) \\ &= 28x^3 + 84x^3 + 189x^2y^2 \\ &= 112x^3 + 189x^2y^2 \\ \frac{\partial f}{\partial y} &= 7x^3(18y) + (4x + 9y^2)(0) \\ &= 126x^3y \end{aligned}$$

■

Quotient Rule

Given $z = g(x,y)/h(x,y), \qquad h(x,y) \neq 0,$

$$\frac{\partial z}{\partial x} = \frac{h(x,y)\cdot(\partial g/\partial x) - g(x,y)\cdot(\partial h/\partial x)}{[h(x,y)]^2} \tag{8.5}$$

$$\frac{\partial z}{\partial y} = \frac{h(x,y)\cdot(\partial g/\partial y) - g(x,y)\cdot(\partial h/\partial y)}{[h(x,y)]^2} \tag{8.6}$$

Problem 8.5. Use the quotient rule to find the first-order partial derivatives for each of the following functions:

(a) $z = (3x + 8y)/(4x + 7y)$

(b) $f(x,y) = \dfrac{x^2 + y^2}{5x + 2y}$

Solution.

(a)

$$\begin{aligned}\frac{\partial z}{\partial x} &= \frac{(4x+7y)(3) - (3x+8y)(4)}{(4x+7y)^2} \\ &= \frac{12x + 21y - 12x - 32y}{(4x+7y)^2} = \frac{-11y}{(4x+7y)^2} \\ \frac{\partial z}{\partial y} &= \frac{(4x+7y)(8) - (3x+8y)(7)}{(4x+7y)^2} \\ &= \frac{32x + 56y - 21x - 56y}{(4x+7y)^2} = \frac{11x}{(4x+7y)^2}\end{aligned}$$

(b)

$$\begin{aligned}\frac{\partial f}{\partial x} &= \frac{(5x+2y)(2x) - (x^2+y^2)(5)}{(5x+2y)^2} \\ &= \frac{10x^2 + 4xy - 5x^2 - 5y^2}{(5x+2y)^2} = \frac{5x^2 + 4xy - 5y^2}{(5x+2y)^2} \\ \frac{\partial f}{\partial y} &= \frac{(5x+2y)(2y) - (x^2+y^2)(2)}{(5x+2y)^2}\end{aligned}$$

$$= \frac{10xy + 4y^2 - 2x^2 - 2y^2}{(5x + 2y)^2} = \frac{2y^2 + 10xy - 2x^2}{(5x + 2y)^2}$$

■

Generalized Power Function Rule

If $z = [g(x, y)]^n$ then,

$$\frac{\partial z}{\partial x} = n[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial x} \tag{8.7}$$

$$\frac{\partial z}{\partial y} = n[g(x, y)]^{n-1} \cdot \frac{\partial g}{\partial y} \tag{8.8}$$

Problem 8.6. Use the generalized power function rule to find the first-order partial derivatives for each of the following functions:

(a) $z = (x^4 + 5y^2)^3$

(b) $f(x, y) = \sqrt{4x + 7y}$

Solution.

(a)

$$\frac{\partial z}{\partial x} = 3(x^4 + 5y^2)^2 \cdot (4x^3) = 12x^3(x^4 + 5y^2)^2$$

$$\frac{\partial z}{\partial y} = 3(x^4 + 5y^2)^2 \cdot (10y) = 30y(x^4 + 5y^2)^2$$

(b)

$$f(x, y) = \sqrt{4x + 7y} = (4x + 7y)^{1/2}$$

$$\frac{\partial f}{\partial x} = \frac{1}{2}(4x + 7y)^{-1/2} \cdot (4) = 2(4x + 7y)^{-1/2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{2}(4x + 7y)^{-1/2} \cdot (7) = \frac{7}{2}(4x + 7y)^{-1/2}$$

■

Natural Exponential Function Rule

If $z = e^{g(x,y)}$ then,

$$\frac{\partial z}{\partial x} = e^{g(x,y)} \cdot \frac{\partial g}{\partial x} \tag{8.9}$$

$$\frac{\partial z}{\partial y} = e^{g(x,y)} \cdot \frac{\partial g}{\partial y} \tag{8.10}$$

Problem 8.7. Use the natural exponential function rule to find the first-order partial derivatives for each of the following functions:

(a) $z = e^{5xy^2}$

(b) $f(x,y) = e^{3xy}$

(c) $z = e^{x^2y^2}$

Solution.

(a)

$$\frac{\partial z}{\partial x} = e^{5xy^2} \cdot (5y^2) = 5y^2 e^{5xy^2}$$

$$\frac{\partial z}{\partial y} = e^{5xy^2} \cdot (10xy) = 10xye^{5xy^2}$$

(b)

$$\frac{\partial f}{\partial x} = e^{3xy} \cdot (3y) = 3ye^{3xy}$$

$$\frac{\partial f}{\partial y} = e^{3xy} \cdot (3x) = 3xe^{3xy}$$

(c)

$$\frac{\partial z}{\partial x} = e^{x^2y^2} \cdot 2xy^2 = 2xy^2 e^{x^2y^2}$$

$$\frac{\partial z}{\partial y} = e^{x^2y^2} \cdot 2x^2y = 2x^2ye^{x^2y^2}$$

■

Natural Logarithmic Function Rule

If $z = \ln|g(x,y)|$ then,

$$\frac{\partial z}{\partial x} = \frac{1}{g(x,y)} \cdot \frac{\partial g}{\partial x} \tag{8.11}$$

$$\frac{\partial z}{\partial y} = \frac{1}{g(x,y)} \cdot \frac{\partial g}{\partial y} \tag{8.12}$$

Problem 8.8. Use the natural logarithmic function rule to find the first-order partial derivatives for each of the following functions:

(a) $z = \ln|4x + y^2|$

(b) $f(x,y) = \ln|x^2 + y^3|$

Solution.

(a)

$$\frac{\partial z}{\partial x} = \frac{1}{4x + y^2} \cdot (4) = \frac{4}{4x + y^2}$$

$$\frac{\partial z}{\partial y} = \frac{1}{4x + y^2} \cdot (2y) = \frac{2y}{4x + y^2}$$

(b)

$$\frac{\partial f}{\partial x} = \frac{1}{x^2 + y^3} \cdot (2x) = \frac{2x}{x^2 + y^3}$$

$$\frac{\partial f}{\partial y} = \frac{1}{x^2 + y^3} \cdot (3y^2) = \frac{3y^2}{x^2 + y^3}$$

■

Problem 8.9. Find the first-order partial derivatives of the following functions

(a) $z = 5x^3 e^{2xy}$

(b) $z = \ln|2x - 5y| \cdot e^{4xy}$

Solution.

(a) Using the product rule and natural exponential function rule,

$$\begin{aligned} z_x &= 5x^3 \cdot 2ye^{2xy} + e^{2xy} \cdot 15x^2 \\ &= 5x^2 e^{2xy}(2xy + 3) \end{aligned}$$

$$\begin{aligned} z_y &= 5x^3 \cdot 2xe^{2xy} + e^{2xy} \cdot 0 \\ &= 10x^4 e^{2xy} \end{aligned}$$

(b) By the product, logarithmic, and exponential function rule,

$$\begin{aligned} z_x &= \ln|2x - 5y| \cdot 4ye^{4xy} + e^{4xy}\left(\frac{1}{2x-5y} \cdot 2\right) \\ &= e^{4xy}\left[4y(\ln|2x-5y|) + \frac{2}{2x-5y}\right] \\ z_y &= \ln|2x - 5y| \cdot 4xe^{4xy} + e^{4xy}\left(\frac{1}{2x-5y} \cdot -5\right) \\ &= e^{4xy}\left[4x(\ln|2x-5y|) - \frac{5}{2x-5y}\right] \end{aligned}$$

■

8.4 Second-Order Partial Derivatives

Given a function $z = f(x, y)$, *the second-order partial derivative* signifies that the function has been differentiated partially with respect to one of the independent variables twice while the other independent variable has been held constant:

$$f_{xx} = (f_x)_x = \frac{\partial z}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2} \qquad f_{yy} = (f_y)_y = \frac{\partial z}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

In words, f_{xx} measures the rate of change of the first-order partial derivative f_x with respect to x while y is held constant. f_{yy} is exactly parallel. The second-order direct partial derivatives measures the rate of change or *slope* of the first-order partial derivative with respect to the axis specified by the independent variable.

The *cross (or mixed)* partial derivative, f_{xy} or f_{yx}, indicates that the primitive function has been first partially differentiated with respect to one independent variable and that the resulting partial derivative has in turn been partially differentiated with respect to the other independent variable:

$$f_{xy} = (f_x)_y = \frac{\partial z}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x} \qquad f_{yx} = (f_y)_x = \frac{\partial z}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

In brief, a cross partial measures the rate of change of a first-order partial derivative with respect to the other independent variable.

Problem 8.10. Find the second-order direct partial derivatives for each of the following functions:

(a) $f(x, y) = 6x^3y^5$

(b) $z = (2x + 5y)(7x - 3y)$

(c) $z = e^{4x-7y}$

Solution.

(a)

$$\begin{aligned} f_x(x, y) &= 18x^2y^5 & \qquad f_y(x, y) = 30x^3y^4 \\ f_{xx}(x, y) &= 36xy^5 & \qquad f_{yy}(x, y) = 120x^3y^3 \end{aligned}$$

(b)

$$\begin{aligned} \frac{\partial z}{\partial x} &= (2x + 5y)(7) + (7x - 3y)(2) \\ &= 28x + 29y \\ \frac{\partial^2 z}{\partial x^2} &= 28 \\ \frac{\partial z}{\partial y} &= (2x + 5y)(-3) + (7x - 3y)(5) \\ &= 29x - 30y \\ \frac{\partial^2 z}{\partial y^2} &= -30 \end{aligned}$$

(c)

$$z_x = e^{4x-7y} \cdot 4 = 4e^{4x-7y} \qquad z_y = e^{4x-7y} \cdot (-7) = -7e^{4x-7y}$$
$$z_{xx} = 4e^{4x-7y} \cdot 4 = 16e^{4x-7y} \qquad z_{yy} = -7e^{4x-7y} \cdot (-7) = 49e^{4x-7y}$$

■

Problem 8.11. For the following functions, estimate the slopes of the first-order partial derivatives with respect to the same principal axis at the points indicated.

(a) $f(x, y) = 5x^4 - 6x^2y^3 - 4y^3$ at (3,2)

(b) $z = \ln|7x - 4y|$ at (2,1)

Solution.

(a)

$$\begin{aligned} f_x(x, y) &= 20x^3 - 12xy^3 \\ f_{xx}(x, y) &= 60x^2 - 12y^3 \\ f_{xx}(3, 2) &= 60(3)^2 - 12(2)^3 \\ &= 444 \end{aligned} \qquad \begin{aligned} f_y(x, y) &= -18x^2y^2 - 12y^2 \\ f_{yy}(x, y) &= -36x^2y - 24y \\ f_{yy}(3, 2) &= -36(3)^2(2) - 24(2) \\ &= -696 \end{aligned}$$

(b)

$$z_x = \frac{1}{7x - 4y} \cdot 7 = \frac{7}{7x - 4y}$$
$$z_y = \frac{1}{7x - 4y} \cdot (-4) = \frac{-4}{7x - 4y}$$

By the quotient rule

$$z_{xx} = \frac{(7x - 4y)(0) - 7(7)}{(7x - 4y)^2} = \frac{-49}{(7x - 4y)^2}$$
$$z_{xx}(2, 1) = \frac{-49}{[7(2) - 4(1)]^2} = \frac{-49}{100} = -0.49$$

$$z_{yy} = \frac{(7x-4y)(0)-(-4)(-4)}{(7x-4y)^2} = \frac{-16}{(7x-4y)^2}$$

$$z_{yy}(2,1) = \frac{-16}{[7(2)-4(1)]^2} = \frac{-16}{100} = -0.16$$

■

Problem 8.12. Find the cross partial derivatives for each of the following functions:

(a) $f(x,y) = 5x^3y^2 - 10x^2y^4$

(b) $z = e^{x^2y^3}$

Solution.

(a)

$$\frac{\partial f}{\partial x} = 15x^2y^2 - 20xy^4$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = 30x^2y - 80xy^3$$

$$\frac{\partial f}{\partial y} = 10x^3y - 40x^2y^3$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = 30x^2y - 80xy^3$$

(b)

$$z_x = e^{x^2y^3} \cdot 2xy^3 = 2xy^3e^{x^2y^3}$$

$$z_{xy} = (z_x)_y = 2xy^3 \cdot 3x^2y^2e^{x^2y^3} + e^{x^2y^3} \cdot 6xy^2 = (6x^3y^5 + 6xy^2)e^{x^2y^3}$$

$$z_y = e^{x^2y^3} \cdot 3x^2y^2 = 3x^2y^2e^{x^2y^3}$$

$$z_{yx} = (z_y)_x = 3x^2y^2 \cdot 2xy^3e^{x^2y^3} + e^{x^2y^3} \cdot 6xy^2 = (6x^3y^5 + 6xy^2)e^{x^2y^3}$$

■

8.5 Optimization of Multivariable Functions

For a multivariable function such as $z = f(x, y)$ at a relative minimum or maximum, three conditions must be met.

1. The first-order partial derivatives must equal zero simultaneously: $f_x(a, b) = f_y(a, b) = 0$. This indicates that at the given point (a, b), called a *critical point*, the function is neither increasing nor decreasing but is at a relative plateau with respect to the principal axes.

2. The second-order direct partial derivatives, when evaluated at the critical point (a, b), must both the positive for a minimum $[f_{xx}(a, b), f_{yy}(a, b) > 0]$ and negative for a maximum $[f_{xx}(a, b), f_{yy}(a, b) < 0]$. This ensures that from a relative plateau at the critical point the function is moving upward in relation to the principal axes in the case of a minimum, and downward in relation to the principal axes in the case of a maximum.

3. The product of the second-order direct partials evaluated at the critical point must exceed the product of the cross partials also evaluated at the critical point. Since $f_{xy} = f_{yx}$, this is written $f_{xx}(a, b) \cdot f_{yy}(a, b) > [f_{xy}(a, b)]^2$ or $f_{xx}(a, b) \cdot f_{yy}(a, b) - [f_{xy}(a, b)]^2 > 0$

Note 8.1. If $f_{xx} \cdot f_{yy} < (f_{xy})^2$

(a) when f_{xx} and f_{yy} have the same sign, the function is at an *inflection point.*

(b) f_{xx} and f_{yy} have different signs, the function is at a *saddle point.*

Note 8.2. If$f_{xx} \cdot f_{yy} = (f_{xy})^2$ the test is inconclusive.

Problem 8.13. Find the critical points and determine whether the function is at a relative minimum or maximum, given

$$f(x, y) = 2x^3 - y^3 - 24x + 75y + 7$$

Solution.

(a) Take the first-order partial derivatives, set them equal to zero, and solve for x and y:

$$\begin{aligned} f_x = 6x^2 - 24 &= 0 \\ 6x^2 &= 24 \\ x^2 &= 4 \\ x &= \pm 2 \\ f_y = -3y^2 + 75 &= 0 \\ -3y^2 &= -75 \\ y^2 &= 25 \\ y &= \pm 5 \end{aligned}$$

with $x = \pm 2, y = \pm 5$, there are four distinct sets of critical points : $(2, 5), (2, -5), (-2, 5)$, and$(-2, -5)$.

(b) To check whether function is at relative minimum or maximum we find second-order direct partials then evaluate them at the given critical points and check the signs:

$$\begin{aligned} f_{xx} &= 12x & f_{yy} &= -6y \\ f_{xx}(2,5) &= 12(2) = 24 > 0 & f_{yy}(2,5) &= -30 < 0 \\ f_{xx}(2,-5) &= 12(2) = 24 > 0 & f_{yy}(2,-5) &= 30 > 0 \\ f_{xx}(-2,5) &= 12(-2) = -24 < 0 & f_{yy}(-2,5) &= -30 < 0 \\ f_{xx}(-2,-5) &= 12(-2) = -24 < 0 & f_{yy}(-2,-5) &= 30 > 0 \end{aligned}$$

Note that at $(2, -5)$ both f_{xx} and f_{yy} are positive.So the function may be at relative minimum at $(2, -5)$. We can also see that at $(-2, 5)$ both f_{xx} and f_{yy} are negative. So the function may be at relative maximum at $(-2, 5)$.

(c) Now we take cross partial derivatives: we have,

$$f_x = 6x^2 - 24 \quad \Rightarrow \quad f_{xy} = (f_x)_y = 0$$

$$f_y = -3y^2 + 75 \quad \Rightarrow \quad f_{yx} = (f_y)_x = 0$$

$$f_{xy} = f_{yx}$$

Note that,

$$f_{xx}(2, -5) \cdot f_{yy}(2, -5) = (24)(30) = 720 > 0 = (0)^2 = [f_{xy}(2, -5)]^2$$

$$\Rightarrow f_{xx}(2,-5) \cdot f_{yy}(2,-5) > [f_{xy}(2,-5)]^2$$

We can now be sure that the function is indeed at a relative minimum at $(2,-5)$. Note that

$$f_{xx}(-2,5) \cdot f_{yy}(-2,5) = (-24)(-30) = 720 > 0 = (0)^2 = [f_{xy}(-2,5)]^2$$

$$\Rightarrow f_{xx}(-2,5) \cdot f_{yy}(-2,5) > [f_{xy}(-2,5)]^2$$

We can now be sure that the function is indeed at a relative maximum at $(-2,5)$. With f_{xx} and f_{yy} of different signs at $(2,5)$ and $(-2,-5)$, $f(2,5)$ and $f(-2,-5)$ are saddle points.

■

Problem 8.14. For the function $f(x,y) = 2x^3 - 4y^2 - 216x + 24y + 7$

(a) Find the critical points where the function is optimized.

(b) Determine whether at these points the function is maximized or minimized.

Solution.

(a) Take the first-order partials and set them equal to zero,

$$f_x = 6x^2 - 216 = 0 \quad (8.13)$$

$$f_y = -8y + 24 = 0 \quad (8.14)$$

Solve for the critical points.

$$\begin{aligned} 6x^2 &= 216 & \qquad -8y &= -24 \\ x^2 &= 36 & y &= 3 \\ x &= \pm 6 \end{aligned}$$

So the critical points are $(6,3)$ and $(-6,3)$.

(b) From (8.13) and (8.14), take the second direct partials,

$$f_{xx} = 12x \qquad f_{yy} = -8$$

Evaluate them at the critical points and note the signs.

$$f_{xx}(6,3) = 12(6) = 72 > 0 \qquad f_{yy}(6,3) = -8 < 0$$
$$f_{xx}(-6,3) = 12(-6) = -72 < 0 \qquad f_{yy}(-6,3) = -8 < 0$$

Then take the cross partials from (8.13) or (8.14),

$$f_{xy} = 0 = f_{yx}$$

Evaluate it at the critical points, and test the third condition.

$$f_{xx}(a,b) \cdot f_{yy}(a,b) > [f_{xy}(a,b)]^2$$

At $(6,3)$ $\qquad 72 \cdot -8 < 0$

At $(-6,3)$ $\qquad -72 \cdot -8 > 0$

With $f_{xx}, f_{yy} < 0$ and $f_{xx}f_{yy} > (f_{xy})^2$ at (-6,3), $f(-6,3)$ is a relative maximum. With f_{xx} and f_{yy} of different signs at $(6,3)$, $f(6,3)$ is a saddle point.

■

8.6 Constrained Optimization and Lagrange Multipliers

Many business and science problems call for optimizing a function subject to a given constraint. Given a function $f(x,y)$ subject to a constraint $g(x,y) = k$ (a constant), a helpful new function can be formed simply by,

(1) Setting the constraint equal to zero.

(2) Multiplying it by λ (the *Lagrange multiplier*).

(3) Adding the product to the original function to obtain

$$F(x,y,\lambda) = f(x,y) + \lambda[g(x,y) - k] \qquad (8.15)$$

Here $F(x, y, \lambda)$ is the *Lagrangian function*, $f(x, y)$ is the original or *objective function*, and $g(x, y) = k$ is the *constraint*. Since the constraint is always set equal to zero, the product $\lambda[g(x, y) - k]$ also equals zero and the addition of the term does not change the value of the objective function. Critical values x_0, y_0, and λ_0, at which the function is optimized, are found by taking the partial derivatives of F with respect to all *three* independent variables, setting them equal to zero, and solving simultaneously:

$$F_x(x, y, \lambda) = 0 \qquad F_y(x, y, \lambda) = 0 \qquad F_\lambda(x, y, \lambda) = 0$$

Problem 8.15. Use Lagrange multipliers minimize the function

$$f(x, y) = 3x^2 + 5xy + 8y^2$$

subject to the constraint $x + y = 48$.

Solution.

(a) Set the constraint equal to zero,

$$x + y - 48 = 0 \qquad \textit{or} \qquad 48 - x - y = 0$$

Multiply it by λ and add it to the objective function to form the Lagrange function F.

$$F = 3x^2 + 5xy + 8y^2 + \lambda(x + y - 48) \tag{8.16}$$

(b) Take the first-order partials, set them equal to zero, and solve simultaneously.

$$F_x = 6x + 5y + \lambda = 0 \tag{8.17}$$
$$F_y = 5x + 16y + \lambda = 0 \tag{8.18}$$
$$F_\lambda = x + y - 48 = 0 \tag{8.19}$$

Subtracting (8.18) from (8.17) we get,

$$x - 11y = 0 \qquad\qquad x = 11y$$

Substituting $x = 11y$ in (8.19),

$$11y + y = 48 \Rightarrow y_0 = 4$$

From which we find,

$$x_0 = 44 \qquad\qquad \lambda_0 = -284$$

Substituting the critical values in (8.16),

$$\begin{aligned} F &= 3(44)^2 + 5(44)(4) + 8(4)^2 + (-284)(44 + 4 - 48) \\ &= 3(1936) + 5(176) + 8(16) - 284(0) = 6816 \end{aligned}$$

■

Note that at the critical points $F = f$.

Problem 8.16. Use Lagrange multipliers maximize the function

$$f(x, y, z) = 3x^2yz$$

subject to the constraint $x + y + z = 32$.

Solution.

(a) Set the constraint equal to zero,

$$x + y + z - 32 = 0.$$

Multiply it by λ and add it to the objective function to form the Lagrange function F.

$$F = 3x^2yz + \lambda(x + y + z - 32) \tag{8.20}$$

(b) Take the first-order partials, set them equal to zero, and solve simultaneously.

$$\begin{aligned} F_x &= 6xyz + \lambda = 0 \qquad (8.21) \\ F_y &= 3x^2z + \lambda = 0 \qquad (8.22) \\ F_y &= 3x^2y + \lambda = 0 \qquad (8.23) \\ F_\lambda &= x + y + z - 32 = 0 \qquad (8.24) \end{aligned}$$

Equating λ's from (8.22) from (8.23) we get,

$$3x^2z = 3x^2y \qquad \Rightarrow y = z$$

Equating λ's from (8.21) from (8.22) we get,

$$3x^2z = 6xyz \qquad \Rightarrow x = 2y$$

Substituting $x = y$ and $x = 2y$ (8.24),

$$2y + y + y - 32 = 0 \Rightarrow 4y = 32 \qquad \Rightarrow y_0 = 8$$

From which we find,

$$x_0 = 16 \qquad z_0 = 8 \qquad \lambda_0 = -6144$$

Substituting the critical values in (8.20),

$$\begin{aligned} F &= 3(16)^2(8)(8) + (-6144)(16 + 8 + 8 - 32) \\ &= 3(16354) = 49,152 \end{aligned}$$

8.7 Total Differentials

The *total differential* measures the change in the dependent variable brought about by a small change in each of the independent variables. Given $z = f(x, y)$, the total differential (dz) is expressed mathematically as

$$dz = z_x \cdot dx + z_y \cdot dy$$

where z_x and z_y are the partial derivatives of z with respect to x and y respectively, and dx and dy represent small changes in x and y.

Problem 8.17. Find the total differentials for the following functions:

(a) $z = x^3 + 7xy + 5y^4$

(b) $f = 15x^4y^5$

(c) $f(x, y, z) = 5x^6y^3z^4$

Solution.

(a) We have total differential $dz = z_x \cdot dx + z_y \cdot dy$

$$\begin{aligned} z &= x^3 + 7xy + 5y^4 \\ z_x &= 3x^2 + 7y \\ z_y &= 7x + 20y^3 \\ dz = (3x^2 + 7y)dx &+ (7x + 20y^3)dy \end{aligned}$$

(b)

$$\begin{aligned} f &= 15x^4y^5 \\ f_x &= 60x^3y^5 \\ f_y &= 75x^4y^4 \\ df = (60x^3y^5)dx &+ (75x^4y^4)dy \end{aligned}$$

(c)

$$\begin{aligned} f &= 5x^6y^3z^4 \\ f_x &= 30x^5y^3z^4 \\ f_y &= 15x^6y^2z^4 \\ f_z &= 20x^6y^3z^3 \\ df = (30x^5y^3z^4)dx &+ (15x^6y^2z^4)dy + (20x^6y^3z^3)dz \end{aligned}$$

■

Problem 8.18. (1) Find the critical points at which the given profit function

$$\pi = 190x - 3x^2 - 5xy - 4y^2 + 235y - 54$$

is maximized and (2)test the second- order conditions.

Solution.

(1) Take the first-order partials, set them equal to zero and solve simultaneously,

$$\pi_x = -6x - 5y + 190 = 0$$

$$\pi_y = -5x - 8y + 235 = 0$$

Solving we get $x = 15$ and $y = 20$. So (15,20) is the critical point.

(2) Note that $\pi_{xx} = -6$, $\pi_{yy} = -8$ and $\pi_{xy} = -5$.

At (15,20), $\pi_{xx} = -6$ and $\pi_{yy} = -8$ (a constant function will remain constant)

Note that $\pi_{xx} \cdot \pi_{yy} = (-6)(-8) = 48 > (-5)^2 = (\pi_{xy})^2$

with $\pi_{xx}, \pi_{yy} < 0$ and $\pi_{xx}\pi_{yy} > (\pi_{xy})^2$, profits are maximized at (15,20).

8.8 Exercises

1. Find the first-order partial derivatives for each of the following functions:

   (a) $z = 13w^3 + 3w^2x^3y^4 - 10x^4 - 11y^3$

   (b) $(2w^5 + 3x^2)(w^3 - 5x^4 + 4y^2)$

   (c) $z = \dfrac{w^3 + y^2}{8w + 4x + 3y}$

   (d) $z = (8x^2 + 3xy^3)^5$

   (e) $z = \ln(5 + 2x^3y^5)$

2. Find the first-order partial derivatives for each of the following functions:

   (a) $z = \dfrac{y}{1 + e^x}$

   (b) $z = \dfrac{7xy}{e^{2x+1}}$

   (c) $z = y(1 + e^x)^{-1}$

   (d) $z = 7xye^{-(2x+1)}$

3. Find the second-order direct partial derivatives for each of the following functions:
   (a) $f(x, y) = 4x^6 - 3x^2y^2 + 5y^4$
   (b) $f(x, y, z) = 10x^3y^2z^4$
4. Find the cross partial derivatives for each of the following functions:
   (a) $f(x, y, z) = x^3y^{-4}z^{-5}$
   (b) $z = \ln(x^2 + 5y)$
5. For the function $f(x, y) = 4x^3 + 6y^2 - 48xy + 9$
   (a) Find the critical points where the function is optimized.
   (b) Determine whether at these points the function is maximized or minimized.
6. Use Lagrange multipliers minimize the function

$$f(x, y, z) = 4xy + 7xz + 9yz$$

   subject to the constraint $xyz = 2016$.
7. (a) Find the critical points at which the given profit function

$$\pi = 266x - 5x^2 - 2xy - 6y^2 + 146y - 104$$

   is maximized.
   (b) Test the second- order conditions.

Chapter 9

More of Integration and Multivariable Calculus

9.1 Integration by Substitution

Up to this stage, we can do simple integration using formulas and simple rules. For more complicated ones, like $\int xe^{x^2}dx$, we have to use some techniques for integration. In general, different forms of the integrand requires different techniques. In this section, we discuss a simple but important technique the *substitution method*. It is the technique in integration that corresponds to the chain rule in differentiation

By expressing the integrand $f(x)$ as a function of u and its derivative du/dx, and integrating with respect to x, we obtain

$$\int f(x)dx = \int \left(u\frac{du}{dx}\right) dx$$

$$\int f(x)dx = \int udu = F(u) + c$$

Steps for the Substitution Method

(1) Define a new variable $u = g(x)$, where $g(x)$ is chosen in such a way that $g'(x)$ "is a factor " of the integrand and that when written in terms of u, the integrand is simpler than when written in terms of x.

(2) Transform the integral with respect to x into an integral with respect to u by replacing $g(x)$ everywhere by u and $g'(x)dx$ by du.

(3) Integrate the resulting function of u.

(4) Substitute back $u = g(x)$ to express the result in terms of x.

Problem 9.1. Determine the integral $\int 24x^3(x^4+9)dx$.

Solution.

(1) Let $u = x^4 + 9$, then $du/dx = 4x^3 \Rightarrow dx = du/4x^3$

(2) Substitute u for $x^4 + 9$ and $du/4x^3$ for dx in the original integrand we get,

$$\int 24x^3(x^4+9)dx = \int 24x^3 \cdot u \cdot \frac{du}{4x^3} = \int 6udu$$

(3) Integrate with respect to u we get,

$$\int 6udu = 6\int udu = 6\left(\frac{u^2}{2}\right) + c = 3u^2 + c$$

(4) Replace u by (x^4+9) we get,

$$\int 24x^3(x^4+9)dx = 3(x^4+9)^2 + c$$

■

Problem 9.2. Use integration by substitution to evaluate the indefinite integral for the following functions:

(a) $\displaystyle\int \frac{56x}{(7x^2+4)^3}dx$

(b) $\displaystyle\int 60x^3\sqrt{5x^4+9}dx$

(c) $\displaystyle\int 8xe^{4x^2}dx$

(d) $\displaystyle\int \frac{\ln x}{x}dx$

(e) $\int \frac{\ln 5x}{x} dx$

Solution.

(a) (1) Let $u = 7x^2 + 4$, then $du/dx = 14x \Rightarrow dx = du/14x$

(2) Substitute u for $7x^2 + 4$ and $du/14x$ for dx in the original integrand we get,

$$\int \frac{56x}{(7x^2+4)^3} dx = \int 56x \cdot \frac{1}{u^3} \cdot \frac{du}{14x} = \int \frac{4}{u^3} du$$

(3) Integrate with respect to u we get,

$$\int \frac{4}{u^3} du = 4 \int u^{-3} du = 4 \left(\frac{u^{-2}}{-2} \right) + c = -2u^{-2} + c$$

(4) Replace u by $(7x^2 + 4)$ we get,

$$\int \frac{56x}{(7x^2+4)^3} dx = -2(7x^2+4)^{-2} + c = \frac{-2}{(7x^2+4)^2} + c$$

(b) (1) Let $u = 5x^4 + 9$, then $du/dx = 20x^3 \Rightarrow dx = du/20x^3$

(2) Substitute u for $5x^4 + 9$ and $du/20x^3$ for dx in the original integrand we get,

$$\int 60x^3 \sqrt{5x^4+9} dx = \int 60x^3 \cdot \sqrt{u} \cdot \frac{du}{20x^3} = \int 3\sqrt{u} du$$

(3) Integrate with respect to u we get,

$$\int 3\sqrt{u} du = 3 \int u^{1/2} du = 3 \left(\frac{2}{3} u^{3/2} \right) + c = 2u^{3/2} + c$$

(4) Replace u by $(5x^4 + 9)$ we get,

$$\int 60x^3 \sqrt{5x^4+9} dx = 2(5x^4+9)^{3/2} + c$$

(c) (1) Let $u = 4x^2$, then $du/dx = 8x \Rightarrow dx = du/8x$

(2) Substitute u for $4x^2$ and $du/8x$ for dx in the original integrand we get,

$$\int 8xe^{4x^2} dx = \int 8x \cdot e^u \cdot \frac{du}{8x} = \int e^u du$$

(3) Integrate with respect to u we get,

$$\int e^u du = e^u + c$$

(4) Replace u by $4x^2$ we get,

$$\int 8xe^{4x^2} dx = e^{4x^2} + c$$

(d) (1) Let $u = \ln x$, then $du/dx = \frac{1}{x} \Rightarrow dx = xdu$

(2) Substitute u for $\ln x$ and xdu for dx in the original integrand we get,

$$\int \frac{\ln x}{x} dx = \int \frac{1}{x} \cdot u \cdot xdu = \int udu$$

(3) Integrate with respect to u we get,

$$\int udu = \frac{u^2}{2} + c$$

(4) Replace u by $\ln x$ we get,

$$\int \frac{\ln x}{x} dx = \frac{(\ln x)^2}{2} + c = \frac{\ln^2 x}{2} + c$$

(e) (1) Let $u = \ln 5x$, then $du/dx = \frac{1}{5x} \cdot 5 = x \Rightarrow dx = xdu$

(2) Substitute u for $\ln 5x$ and xdu for dx in the original integrand we get,

$$\int \frac{\ln 5x}{x} dx = \int \frac{1}{x} \cdot u \cdot xdu = \int udu$$

(3) Integrate with respect to u we get,

$$\int udu = \frac{u^2}{2} + c$$

(4) Replace u by $\ln x$ we get,

$$\int \frac{\ln 5x}{x} dx = \frac{(\ln 5x)^2}{2} + c = \frac{\ln^2 5x}{2} + c$$

9.2 Integration by Parts

The technique in integration that corresponds to the product rule in differentiation is called *integration by parts*. When an integrand which is a product or quotient of two differentiable functions of x cannot be expressed as a constant multiple of udu/dx, integration by parts is frequently helpful.

We have the product rule,

$$\frac{d}{dx}[f(x)\cdot g(x)] = f(x)\cdot g'(x) + g(x)\cdot f'(x)$$

Taking the integral of the derivative we get,

$$f(x)\cdot g(x) = \int [f(x)\cdot g'(x)]dx + \int [g(x)\cdot f'(x)]dx$$

From above equation we get,

$$\int [f(x)\cdot g'(x)]dx = f(x)\cdot g(x) - \int [g(x)\cdot f'(x)]dx \tag{9.1}$$

However, to keep things simpler, let us replace $f(x)$ and $g(x)$ with u and v, respectively. Doing this, we can rewrite the expression as

$$\int u\left(\frac{dv}{dx}\right)dx = uv - \int v\left(\frac{du}{dx}\right)dx$$

The above equation we can write as

$$\int udv = uv - \int vdu \tag{9.2}$$

Problem 9.3. Use integration by parts to evaluate the indefinite integral for the following functions

(a) $\int 9x(x+5)^2dx$

(b) $\int \frac{12x}{(x+9)^5}dx$

(c) $\int 15x\sqrt{8+x}dx$

(d) $\int (x+3)e^x dx$

(e) $\int x^3 \ln 2x dx$

(f) $\int \ln x^4 dx$

Solution.

(a) In integration by parts the key thing is to choose u and dv correctly.

Let $u = 9x$ and $dv = (x+5)^2 dx$. Then $du = 9dx$ and $v = \dfrac{(x+5)^3}{3} + c_1$
From (9.2),

$$\int u dv = uv - \int v du$$

$$\begin{aligned}
\int 9x(x+5)^2 dx &= uv - \int v du \\
&= 9x\left[\frac{(x+5)^3}{3} + c_1\right] - \int \left[\frac{(x+5)^3}{3} + c_1\right](9)dx \\
&= 3x(x+5)^3 + 9c_1 x - \int [3(x+5)^3 + 9c_1]dx \\
&= 3x(x+5)^3 + 9c_1 x - 3\frac{(x+5)^4}{4} - 9c_1 x + c \\
&= 3x(x+5)^3 - \frac{3}{4}(x+5)^4 + c
\end{aligned}$$

Thus we get $\int 9x(x+5)^2 dx = 3x(x+5)^3 - \frac{3}{4}(x+5)^4 + c.$

(b) Let $u = 12x$ and $dv = \dfrac{1}{(x+9)^5}dx = (x+9)^{-5}dx$. Then $du = 12dx$ and $v = -\dfrac{(x+9)^{-4}}{4} + c_1$. Substituting in (9.2) we get,

$$\begin{aligned}
\int \frac{12x}{(x+9)^5} dx &= uv - \int v du \\
&= 12x\left[-\frac{(x+9)^{-4}}{4} + c_1\right] - \int \left[-\frac{(x+9)^{-4}}{4} + c_1\right](12)dx \\
&= -3x(x+9)^{-4} + 12c_1 x - \int [-3(x+9)^{-4} + 12c_1]dx \\
&= -3x(x+9)^{-4} + 12c_1 x + \frac{3(x+9)^{-3}}{-3} - 12c_1 x + c
\end{aligned}$$

$$\begin{aligned} &= -3x(x+9)^{-4} + 12c_1x - (x+9)^{-3} - 12c_1x + c \\ &= -3x(x+9)^{-4} - (x+9)^{-3} + c \\ &= -\frac{3x}{(x+9)^4} - \frac{1}{(x+9)^3} + c \end{aligned}$$

Thus we get $\int \frac{12x}{(x+9)^5}dx = -\frac{3x}{(x+9)^4} - \frac{1}{(x+9)^3} + c$

(c) Let $u = 15x$ and $dv = \sqrt{8+x}dx = (8+x)^{1/2}dx$. Then $du = 15dx$ and $v = \frac{2}{3}(8+x)^{3/2}$. Substituting in (9.2) we get,

$$\begin{aligned} \int 15x\sqrt{8+x}dx &= uv - \int vdu \\ &= 15x\left[\frac{2}{3}(8+x)^{3/2}\right] - \int \left[\frac{2}{3}(8+x)^{3/2}\right](15)dx \\ &= 10x(8+x)^{3/2} - \int 10(8+x)^{3/2}dx \\ &= 10x(8+x)^{3/2} - 10\int (8+x)^{3/2}dx \\ &= 10x(8+x)^{3/2} - 10\left[\frac{2}{5}(8+x)^{5/2}\right] + c \\ &= 10x(8+x)^{3/2} - 4(8+x)^{5/2} + c \end{aligned}$$

Thus we get $\int 15x\sqrt{8+x}dx = 10x(8+x)^{3/2} - 4(8+x)^{5/2} + c$

(d) Let $u = x+3$ and $dv = e^x dx$. Then $du = 1dx$ and $v = e^x$. Substituting in (9.2) we get,

$$\begin{aligned} \int (x+3)e^x dx &= uv - \int vdu \\ &= (x+3)e^x - \int e^x \cdot 1dx \\ &= (x+3)e^x - \int e^x dx \\ &= (x+3)e^x - e^x + c \end{aligned}$$

Thus we get $\int (x+3)e^x dx = (x+3)e^x - e^x + c$

(e) Here let $u = \ln 2x$ and $dv = x^3 dx$. Then $du = \frac{1}{x}dx$ and $v = \frac{x^4}{4}$. Substituting in (9.2) we get,

$$\begin{aligned}
\int x^3 \ln 2x dx &= uv - \int v du \\
&= \ln 2x \cdot \frac{x^4}{4} - \int \frac{x^4}{4} \cdot \frac{1}{x} dx \\
&= \frac{x^4}{4} \ln 2x - \int \frac{x^3}{4} dx \\
&= \frac{x^4}{4} \ln 2x - \frac{1}{4} \int x^3 dx \\
&= \frac{x^4}{4} \ln 2x - \frac{1}{4} \left(\frac{x^4}{4} \right) + c \\
&= \frac{x^4}{4} \ln 2x - \frac{1}{16} x^4 + c
\end{aligned}$$

Thus we get $\int x^3 \ln 2x dx = \frac{x^4}{4} \ln 2x - \frac{1}{16} x^4 + c$

(f) Here let $u = \ln x^4$ and $dv = 1dx$. Then $du = \frac{1}{x^4}(4x^3)dx = \frac{4}{x}dx$ and $v = x$. Substituting in (9.2) we get,

$$\begin{aligned}
\int \ln x^4 dx &= uv - \int v du \\
&= \ln x^4 \cdot x - \int \frac{4}{x} \cdot x dx \\
&= x \ln x^4 - \int 4 dx \\
&= x \ln x^4 - 4 \int dx \\
&= x \ln x^4 - 4x + c
\end{aligned}$$

Thus we get $\int \ln x^4 dx = x \ln x^4 - 4x + c$

■

9.3 Improper Integrals

Our original discussion of the definite integral $\int_a^b f(x)dx$ assumed that the interval $a \le x \le b$ was of finite length and that f was continuous. Integrals that arise in applications don't necessarily have these nice properties. In this section we investigate a class of integrals, called *improper integrals*, in which one limit of integration is infinite. So

$$\int_a^\infty f(x)dx, \qquad \int_{-\infty}^b f(x)dx$$

are improper integrals . Note that

$$\begin{aligned}\int_a^\infty f(x)dx &= \lim_{b\to\infty}\int_a^b f(x)dx\\ \int_\infty^b f(x)dx &= \lim_{a\to-\infty}\int_a^b f(x)dx\end{aligned}$$

If the limit in either case exists, the improper integral is said to *converge*. If the limit does not exist, the improper integral *diverges* and meaningless.

Problem 9.4. Evaluate the following improper integrals whenever they are convergent:

(a) $\displaystyle\int_1^\infty \frac{4}{x^2}dx$

(b) $\displaystyle\int_1^\infty \frac{5}{\sqrt{x}}dx$

(c) $\displaystyle\int_5^\infty \frac{dx}{(x-1)^2}$

(d) $\displaystyle\int_0^\infty 21e^{-7x}dx$

Solution.

(a)

$$\begin{aligned}\int_1^\infty \frac{4}{x^2}dx &= \lim_{b\to\infty}\int_1^b \frac{4}{x^2}dx = \lim_{b\to\infty}\left[\frac{-4}{x}\right]_1^b\\ &= \lim_{b\to\infty}\left[\frac{-4}{b}-\left(\frac{-4}{1}\right)\right]\\ &= \lim_{b\to\infty}\left(\frac{-4}{b}+4\right) = 4\end{aligned}$$

because $b \to \infty, (-4/b) \to 0$. Hence the improper integral is convergent and

$$\int_1^\infty \frac{4}{x^2}dx = 4$$

(b)

$$\begin{aligned}
\int_1^\infty \frac{5}{\sqrt{x}} &= \lim_{b\to\infty}\int_1^b \frac{5}{\sqrt{x}}dx = \lim_{b\to\infty}\int_1^b 5x^{-1/2}dx \\
&= \lim_{b\to\infty}\left[5\frac{x^{1/2}}{1/2}\right]_1^b \\
&= \lim_{b\to\infty}\left[10x^{1/2}\right]_1^b \\
&= \lim_{b\to\infty}\left[10\sqrt{x}\right]_1^b \\
&= \lim_{b\to\infty}\left[10\sqrt{b}-10\right]
\end{aligned}$$

because $b \to \infty, (10\sqrt{b}-10)$ increases without bound. So the integral has no limit and is divergent.

(c)

$$\begin{aligned}
\int_5^\infty \frac{dx}{(x-1)^2} &= \lim_{b\to\infty}\int_5^b \frac{dx}{(x-1)^2} = \lim_{b\to\infty}\int_5^b (x-1)^{-2}dx \\
&= \lim_{b\to\infty}\left[\frac{(x-1)^{-1}}{-1}\right]_5^b \\
&= \lim_{b\to\infty}\left[\frac{-1}{(x-1)}\right]_5^b \\
&= \lim_{b\to\infty}\left[\frac{-1}{(b-1)}+\frac{1}{4}\right] \\
&= \lim_{b\to\infty}\left[\frac{-1}{(b-1)}\right]+\lim_{b\to\infty}\frac{1}{4} \\
&= \frac{1}{4}
\end{aligned}$$

because $b \to \infty, -1/(b-1) \to 0$. Hence the improper integral is convergent and

$$\int_5^\infty \frac{dx}{(x-1)^2} = \frac{1}{4}$$

(d)

$$\begin{aligned}\int_0^\infty 21e^{-7x}dx &= \lim_{b\to\infty}\int_0^b 21e^{-7x}dx \\ &= \lim_{b\to\infty}\left[\frac{21e^{-7x}}{-7}\right]_0^b \\ &= \lim_{b\to\infty}\left[-3e^{-7x}\right]_0^b \\ &= \lim_{b\to\infty}\left[-3e^{-7b}+3e^{-7(0)}\right] \\ &= \lim_{b\to\infty}\left[-3e^{-7b}+3\right]=3\end{aligned}$$

because $b\to\infty, -3e^{-7b}\to 0$. Hence the improper integral is convergent and

$$\int_0^\infty 21e^{-7x}dx = 3$$

■

Problem 9.5. Evaluate the following improper integrals whenever they are convergent:

(a) $\int_1^\infty \frac{8}{x}dx$

(b) $\int_{-\infty}^0 28e^{4x}dx$

Solution.

(a)

$$\begin{aligned}\int_1^\infty \frac{8}{x}dx &= \lim_{b\to\infty}\int_1^b \frac{8}{x}dx \\ &= \lim_{b\to\infty}[8\ln x]_1^b \\ &= \lim_{b\to\infty}[8\ln b - 8\ln 1] \\ &= \lim_{b\to\infty}(8\ln b) \qquad \textit{Since} \quad \ln 1 = 0\end{aligned}$$

As $b\to\infty, 8\ln b\to\infty$, so the improper integral diverges and has no definite value.

(b)

$$\begin{aligned}\int_{-\infty}^{0} 28e^{4x}dx &= \lim_{b\to-\infty}\int_{b}^{0} 28e^{4x}dx \\ &= \lim_{b\to-\infty}\left[\frac{28e^{4x}}{4}\right]_b^0 \\ &= \lim_{b\to-\infty}[7e^{4x}]_b^0 \\ &= \lim_{b\to-\infty}[7e^{4(0)} - 7e^{4b}] \\ &= \lim_{b\to-\infty}[7 - 7e^{4b}] = 7\end{aligned}$$

because $b \to -\infty, -7e^{4b} \to 0$. Hence the improper integral is convergent and

$$\int_{-\infty}^{0} 28e^{4x}dx = 7$$

■

9.4 L'Hôpital's Rule

We use L'hôpital's rule if the limit of a function $f(x) = g(x)/h(x)$ as $x \to a$ can not be evaluated, such as (1) when both numerator and denominator approach zero, giving rise to the indeterminate form $\frac{0}{0}$ or (2) when both numerator and denominator approach infinity, giving rise to the indeterminate form ∞/∞.

L'hôpital's rule states that

$$\lim_{x\to a}\frac{g(x)}{h(x)} = \lim_{x\to a}\frac{g'(x)}{h'(x)} \tag{9.3}$$

Problem 9.6. Use L'hôpital's rule to evaluate the following limits:

(a) $\lim_{x\to 3}\frac{x-3}{9-x^2}$

(b) $\lim_{x\to\infty}\frac{5x^2-2x}{3x^2+3x}$

(c) $\lim_{x\to\infty}\frac{x-7}{e^x}$

(d) $\lim_{x\to\infty}\frac{1-e^{1/x}}{1/x}$

Solution.

(a) As $x \to 3$, $(x-3)$ and $(9-x^2) \to 0$, giving rise to the indeterminate form $0/0$. So we apply L'hôpital's rule. Differentiating numerator and denominator separately,

$$\lim_{x\to 3} \frac{x-3}{9-x^2} = \lim_{x\to 3} \left(\frac{1}{-2x}\right) = -\frac{1}{6}$$

(b) As $x \to \infty$ both $(5x^2-2)$ and $(3x^2+3x) \to \infty$, giving rise to the indeterminate form ∞/∞. So we apply L'hôpital's rule. Differentiating numerator and denominator separately,

$$\lim_{x\to\infty} \frac{5x^2-2x}{3x^2+3x} = \lim_{x\to\infty} \frac{10x-2}{6x+3}$$

Note that, as $x \to \infty$, both $(10x-2)$ and $(6x+3) \to \infty$, so we again apply L'hôpital's rule. Therefore

$$\lim_{x\to\infty} \frac{10x-2}{6x+3} = \lim_{x\to\infty} \frac{10}{6} = \frac{10}{6} = \frac{5}{3}$$

Thus we get,

$$\lim_{x\to\infty} \frac{5x^2-2x}{3x^2+3x} = \frac{5}{3}$$

(c) As $x \to \infty$, both $(x-7)$ and e^x tend to ∞, giving rise to the indeterminate form ∞/∞ . So we apply L'hôpital's rule. Differentiating numerator and denominator separately,

$$\lim_{x\to\infty} \frac{x-7}{e^x} = \lim_{x\to\infty} \frac{1}{e^x} = \frac{1}{\infty} = 0$$

(d) As $x \to \infty$, both $1-e^{1/x}$ and $(1/x) \to 0$, giving rise to the indeterminate form $0/0$. So we apply L'hôpital's rule. Differentiating numerator and denominator separately,

$$\lim_{x\to\infty} \frac{1-e^{1/x}}{1/x} = \lim_{x\to\infty} \frac{-e^{1/x}(-1/x^2)}{-(1/x^2)} = \lim_{x\to\infty} (-e^{1/x}) = -e^0 = -1$$

■

9.5 Double Integrals

If $z = f(x, y)$ and R_1 is the region in the xy plane bounded by $y = g(x)$ and $y = h(x)$ between $x = a$ and $x = b$. Then the volume of the solid bounded by $f(x, y)$ over the region R_1 is defined by the *double* or *iterated integral*

$$\int\int_{R_1} f(x, y)dydx = \int_a^b \left[\int_{g(x)}^{h(x)} f(x, y)dy\right] dx \tag{9.4}$$

If $z = f(x, y)$ and R_2 is the region in the xy plane bounded by $y = c$ and $y = d$ between $x = g(y)$ and $x = h(y)$. Then the volume of the solid bounded by $f(x, y)$ over the region R_2 is

$$\int\int_{R_2} f(x, y)dxdy = \int_c^d \left[\int_{g(y)}^{h(y)} f(x, y)dx\right] dy \tag{9.5}$$

To compute a double integral such as (9.4), we first compute the inner definite integral

$$\int_{g(x)}^{h(x)} f(x, y)dy$$

taking the antiderivative of f with respect to y by treating x as a constant. This result in a function of x alone which we then simply integrate over the limits $x = a$ to $x = b$. The procedure for the double integral in (9.5) is perfectly analogous.

Problem 9.7. Calculate the following double integrals:

(a) $\int_1^2 \int_2^3 (6x + 2y)dydx$

(b) $\int_0^1 \int_1^2 60x^2y^3dydx$

(c) $\int_0^1 \int_x^{\sqrt{x}} 240xy^3dydx$

(d) $\int_1^2 \int_y^{3y} 12xydxdy$

Solution.

(a) Note that,

$$\int_1^2 \int_2^3 (6x+2y)dydx = \int_1^2 \left(\int_2^3 (6x+2y)dy \right) dx$$

Computing the inner integration with respect to y while holding x constant,

$$\begin{aligned} \int_1^2 \int_2^3 (6x+2y)dydx &= \int_1^2 \left[\int_2^3 (6xy+y^2)\Big|_{y=2}^{y=3} \right] dx \\ &= \int_1^2 [(18x+9)-(12x+4)]\, dx \\ &= \int_1^2 (6x+5)dx \\ &= (3x^2+5x)\Big|_1^2 \\ &= (12+10)-(3+5) = 14 \end{aligned}$$

So,

$$\int_1^2 \int_2^3 (6x+2y)dydx = 14$$

(b) Note that,

$$\int_0^1 \int_1^2 60x^2y^3dydx = \int_0^1 \left(\int_1^2 60x^2y^3dy \right) dx$$

Computing the inner integration with respect to y while holding x constant,

$$\begin{aligned} \int_0^1 \int_1^2 60x^2y^3dydx &= \int_0^1 \left[\int_1^2 (15x^2y^4)\Big|_{y=1}^{y=2} \right] dx \\ &= \int_0^1 (240x^2-15x^2)dx \\ &= \int_0^1 225x^2dx \\ &= 75x^3\Big|_0^1 = 75 \end{aligned}$$

So, $$\int_0^1 \int_1^2 60x^2y^3dydx = 75.$$

(c) Note that,

$$\int_0^1 \int_x^{\sqrt{x}} 240xy^3 dydx = \int_0^1 \left(\int_x^{\sqrt{x}} 240xy^3 dy \right) dx$$

Computing the inner integration with respect to y while holding x constant,

$$\begin{aligned}
\int_0^1 \int_x^{\sqrt{x}} 240xy^3 dydx &= \int_0^1 \left[(60xy^4)\Big|_{y=x}^{y=\sqrt{x}} \right] dx \\
&= \int_0^1 [60x(\sqrt{x})^4 - 60x(x)^4] dx \\
&= \int_0^1 (60x^3 - 60x^5) dx \\
&= (15x^4 - 10x^6)\Big|_0^1 \\
&= (15 - 10) - (0 - 0) = 5
\end{aligned}$$

(d) Note that,

$$\int_1^2 \int_y^{3y} 12xydxdy = \int_1^2 \left(\int_y^{3y} 12xydx \right) dy$$

Computing the inner integration with respect to x while holding y constant,

$$\begin{aligned}
\int_1^2 \int_y^{3y} 12xydxdy &= \int_1^2 \left[(6x^2y)\Big|_{x=y}^{x=3y} \right] dy \\
&= \int_0^1 [6(3y)^2y - 6(y)^2y] dy \\
&= \int_1^2 (54y^3 - 6y^3) dy \\
&= \int_1^2 48y^3 dy \\
&= (12y^4)\Big|_1^2 \\
&= 12(2)^4 - 12 = 192 - 12 = 180
\end{aligned}$$

So the value of the double integral $\int_1^2 \int_y^{3y} 12xydxdy$ is 180.

■

9.6 Approximating Definite Integrals

Given an interval $a \leq x \leq b$ divided in to n equal subintervals, each of length $\Delta x = (b-a)/n$, in which the end points of the subintervals are denoted by $g_0, g_1, \ldots g_n$ and the midpoints by $h_1, h_2, \ldots h_n$ we have,

Rectangular Rule

$$\begin{aligned}\int_a^b f(x)dx &\approx f(h_1)\Delta x + f(h_1)\Delta x + \cdots + f(h_{n-1})\Delta x + f(h_n)\Delta x \\ &= [f(h_1) + f(h_2) + \cdots + f(h_{n-1}) + f(h_n)]\Delta x \qquad (9.6)\end{aligned}$$

Trapezoidal Rule

$$\int_a^b f(x)dx \approx [f(g_0) + 2f(g_1) + 2f(g_2) + \cdots + 2f(g_{n-1}) + f(g_n)]\frac{\Delta x}{2} \quad (9.7)$$

Simpson's Rule

$$\begin{aligned}\int_a^b f(x)dx &\approx [f(g_0) + 4f(h_1) + 2f(g_1) + 4f(h_2) + 2f(g_2) + 4f(h_3) \\ &\quad + 2f(g_3) + \cdots + 2f(g_{n-1}) + 4f(h_n) + f(g_n)]\frac{\Delta x}{6} \qquad (9.8)\end{aligned}$$

Problem 9.8. Approximate the value of the definite integral $\int_6^8 (x-3)^2 dx$ with the (1) Rectangle Rule, (2) Trapezoidal rule, (3) Simpson's rule for $n = 5$. Also check the accuracy of the answers by using integration.

Solution. (1) The given interval is $[6, 8]$ and $n = 5$, So

$$\Delta x = \frac{b-a}{n} = \frac{8-6}{5} = \frac{2}{5} = 0.4$$

So the midpoints h_i are 6.2, 6.6 , 7.0, 7.4 and 7.8

Taking the Rectangle rule from (9.6) and substituting,

$$\int_a^b f(x)dx \approx [f(h_1) + f(h_2) + \cdots + f(h_{n-1}) + f(h_n)]\Delta x$$

$$\begin{aligned}\int_6^8 (x-3)^2 dx &\approx [f(6.2)+f(6.6)+f(7.0)+f(7.4)+f(7.8)]\Delta x\\ &\approx [(6.2-3)^2+(6.6-3)^2+(7-3)^2+(7.4-3)^2+(7.8-3)^2](0.4)\\ &\approx [(3.2)^2+(3.6)^2+(4)^2+(4.4)^2+(4.8)^2](0.4)\\ &\approx (10.24+12.96+16+19.36+23.04)(0.4)\\ &\approx 32.64\end{aligned}$$

(2) The end points g_i are 6, 6.4 , 6.8, 7.2, 7.6 and 8.0

Taking the Trapezoidal rule from (9.7) and substituting,

$$\int_a^b f(x)dx \approx [f(g_0)+2f(g_1)+\cdots+2f(g_{n-1})+f(g_n)]\frac{\Delta x}{2}$$

$$\begin{aligned}\int_6^8 (x-3)^2 dx &\approx [f(6)+2f(6.4)+2f(6.8)+2f(7.2)+2f(7.6)+f(8)]\frac{(0.4)}{2}\\ &\approx [(6-3)^2+2(6.4-3)^2+2(6.8-3)^2+2(7.2-3)^2\\ &\quad +2(7.6-3)^2+(8-3)^2](0.2)\\ &\approx [(3)^2+2(3.4)^2+2(3.8)^2+2(4.2)^2+2(4.6)^2+(5)^2](0.2)\\ &\approx (9+23.12+28.88+35.28+42.32+25)(0.2)\\ &\approx 32.72\end{aligned}$$

(3) Taking the Simpson's rule from (9.8) and substituting,

$$\begin{aligned}\int_a^b f(x)dx &\approx [f(g_0)+4f(h_1)+2f(g_1)+4f(h_2)+2f(g_2)+4f(h_3)\\ &\quad +2f(g_3)+\cdots+2f(g_{n-1})+4f(h_n)+f(g_n)]\frac{\Delta x}{6}\\ \int_6^8 (x-3)^2 dx &\approx [f(6)+4f(6.2)+2f(6.4)+4f(6.6)+2f(6.8)+4f(7)+2f(7.2)\\ &\quad 4f(7.4)+2f(7.6)+4f(7.8)+f(8)]\frac{0.4}{6}\\ &\approx [(6-3)^2+4(6.2-3)^2+2(6.4-3)^2+4(6.6-3)^2+2(6.8-3)^2\\ &\quad +4(7-3)^2+2(7.2-3)^2+4(7.4-3)^2+2(7.6-3)^2+4(7.8-3)^2\\ &\quad +(8-3)^2](0.0667)\end{aligned}$$

$$\approx [(3)^2 + 4(3.2)^2 + 2(3.4)^2 + 4(3.6)^2 + 2(3.8)^2 + 4(4)^2 + 2(4.2)^2 + 4(4.4)^2 + 2(4.6)^2 + 4(4.8)^2 + (5)^2](0.0667)$$
$$\approx 32.683$$

Calculating the definite integral for comparison,

$$\int_6^8 (x-3)^2 dx = \left[\frac{(x-3)^3}{3}\right]_6^8 = \frac{1}{3}[5^3 - 3^3] = 32.6667$$

■

9.7 Differential Equations

A *differential equation* is an equation expressing a relationship between a function $y = f(t)$ and one or more of its derivatives.

Example 9.1. Following are some examples of differential equations:

(1) $\frac{dy}{dt} = 2t^2 + 5$

(2) $y' = 6y$

(3) $y'' - 3y' + 5 = 0$

The *solution of a differential equation* is any function, without derivative or differential, which is defined over an interval and satisfies the differential equation for all t in the interval.

Example 9.2. To solve the differential equation $dy/dt = 8t + 5$ for all the functions $y(t)$ which satisfy the equation, simply integrate both sides to find the antiderivatives.

$$\int \frac{dy}{dt} dt = \int (8t + 5) dt$$

$$y + c_1 = 4t^2 + 5t + c_2$$

$$y = 4t^2 + 5t + c \qquad \text{where } c = c_2 - c_1$$

This is called a general solution, indicating that when c is unspecified, a differential equation has an infinite number of possible solutions. If c can be specified, we have a particular solution and it alone of all the possible solutions is relevant.

Example 9.3. A frequently encountered differential equation is one in which the derivative is expressed as a constant multiple of the function itself;

$$y' = ky$$

This type of differential equation cannot be solved with the method above. Recalling, however, that given

$$y = Ae^{kt} \qquad\qquad y' = k \cdot Ae^{kt}$$

By substituting y for Ae^{kt} in y', we have

$$y' = ky$$

Hence a solution for an equation in the familiar form $y' = ky$ is

$$y = Ae^{kt} \qquad A, \;\; k \;\; constants.$$

Problem 9.9. Find (1) the general and (2) the particular solution for each of the following differential equations:

(a) $y'(t) = 9t^2 - 14t + 34, \; y(0) = 5$

(b) $\dfrac{dy}{dt} = 20(t-6)^3, \;\; y(7) = 8$

Solution .

(a) (1) Integrate both sides, ignoring the constant of integration until final step.

$$\begin{aligned} \int y'(t)dt &= \int (9t^2 - 14t + 34)dt \\ y &= 9 \cdot \frac{t^3}{3} - 14 \cdot \frac{t^2}{2} + 34t + c \\ y &= 3t^3 - 7t^2 + 34t + c \end{aligned}$$

(2) Given $y(0) = 5$. This condition is known as *initial condition.* $y(0) = 5$ means at $t = 0$, $y = 5$. Substitute this in the general solution we get,

$$5 = 3(0)^3 - 7(0)^2 + 34(0) + c \qquad \Rightarrow \qquad c = 5$$

So the solution (particular solution) is $y = 3t^3 - 7t^2 + 34t + 5$

(b) (1) Integrate both sides,

$$\begin{aligned} \int \frac{dy}{dt} dt &= \int 20(t-6)^3 dt \\ y &= 20 \cdot \frac{(t-6)^4}{4} + c \\ y &= 5(t-6)^4 + c \end{aligned}$$

(2) Using the information $y(7) = 8$, called a *boundary condition* whenever the value of y is given at a point other than $t = 0$,

$$8 = 5(7-6)^4 + c = 5 + c \qquad \Rightarrow \quad c = 3$$

So the solution (particular solution) is $y = 5(t-6)^4 + 3$

■

9.8 Separation of Variables

A differential equation in the form

$$\frac{dy}{dt} = \frac{f(t)}{g(y)} \tag{9.9}$$

is *separable* if it can be rearranged by multiplication so that all terms with y including dy are on one side of the equal sign and all terms with t including dt are on the other, such as

$$g(y)dy = f(t)dt$$

where $g(y)$ is a function of y alone and $f(t)$ is a function of t alone. A solution for such an equation is then obtainable by simply integrating each side of the equation with respect to its individual variable:

$$\int g(y)dy = \int f(t)dt$$

Problem 9.10. Find the general solution for each of the following differential equations by separating the variables:

(a) $\dfrac{dy}{dt} = \dfrac{15t^2}{4y^3}$

(b) $2y\dfrac{dy}{dt} = 6t^2$

(c) $y' = t^2e^{3y}$

Solution.

(a) (1) Treating dy/dt as a ratio of differentials, separate the variables (by multiplying both sides of the equation by $4y^3dt$), integrate and solve algebraically for y,

$$\begin{aligned}
\int 4y^3dy &= \int 15t^2dt \\
4 \cdot \frac{y^4}{4} + c_1 &= 15 \cdot \frac{t^3}{3} + c_2 \\
y^4 + c_1 &= 5t^3 + c_2 \\
y^4 &= 5t^3 + c \qquad \textit{where } c = c_2 - c_1 \\
y &= \pm\sqrt[4]{5t^3 + c}
\end{aligned}$$

(2) Check the answer by differentiating and substituting.

$$\begin{aligned}
\frac{dy}{dt} = \frac{d}{dt}(5t^3 + c)^{1/4} &= \frac{1}{4}(5t^3 + c)^{-3/4} \cdot 15t^2 \\
&= \frac{15t^2}{4} \cdot \frac{1}{(5t^3 + c)^{3/4}} \\
&= \frac{15t^2}{4} \cdot \frac{1}{[(5t^3 + c)^{1/4}]^3} \\
&= \frac{15t^2}{4} \cdot \frac{1}{y^3} \\
&= \frac{15t^2}{4y^3}
\end{aligned}$$

(b) The given differential equation is $2y\dfrac{dy}{dt} = 6t^2$, by separating variables we get,

$$2ydy = 6t^2dt$$

Integrating both sides we get,

$$\begin{aligned}
\int 2y dy &= \int 6t^2 dt \\
y^2 &= 2t^3 + c \\
y &= \pm\sqrt{2t^3 + c}
\end{aligned}$$

(c) The given differential equation is $\frac{dy}{dt} = t^2 e^{3y}$, by separating variables we get,

$$\frac{dy}{e^{3y}} = t^2 dt$$

Integrating both sides we get,

$$\begin{aligned}
\int \frac{dy}{e^{3y}} &= \int t^2 dt \\
\int e^{-3y} dy &= \int t^2 dt \\
\frac{e^{-3y}}{-3} &= \frac{t^3}{3} + c \\
e^{-3y} &= -t^3 + k \qquad \textit{where} \;\; k = -3c
\end{aligned}$$

Take natural logarithm on both sides (recall $\ln e^{f(x)} = f(x)$) we get,

$$\begin{aligned}
-3y &= \ln(-t^3 + k) \\
y &= -\frac{1}{3}\ln(k - t^3)
\end{aligned}$$

■

9.9 Practical Applications

Integral calculus and differential equations are used in business and all the sciences. Integration is used to evaluate the present value of cash flows, for instance; differential equations to study various forms of population growth under different circumstances.

Example 9.4. If the present value P under continuous compounding of a sum of money to be received in the future is $P = Ae^{-rt}$, the present value of a *cash flow* (money to be received each year for n years) is given by the integral

$$\begin{aligned}
P_n &= \int_0^n Ae^{-rt}dt = A\int_0^n e^{-rt}dt \\
&= A\left[-\frac{e^{-rt}}{r}\right]_0^n = -\frac{A}{r}\left[e^{-rt}\right]_0^n \\
&= -\frac{A}{r}(e^{-rn} - e^{-r(0)}) = -\frac{A}{r}(e^{-rn} - 1) \\
&= \frac{A}{r}(1 - e^{-rn})
\end{aligned}$$

$$\therefore P_n = \frac{A}{r}(1 - e^{-rn}) \tag{9.10}$$

Example 9.5. The growth of populations facing factors that inhibit growth over time, such as the scarcity of food or water, is given by the differential equation, called a limited *growth function*

$$\frac{dy}{dt} = k(M - y) \tag{9.11}$$

where M is the maximum population size, y is the current population size, and k is the growth rate.

Problem 9.11. Find the present value of \$ 5,000 to be paid each year for 4 years when the interest rate is 8% compounded continuously

Solution. From (9.10) we have,

$$P_n = \frac{A}{r}(1 - e^{-rn})$$

Here $A = 5,000,\ r = 8\% = 0.08,\ t = 4$. Substituting in above equation we get,

$$\begin{aligned}
P_n &= \frac{5,000}{0.08}(1 - e^{-(0.08)(4)}) \\
&= 62,500(1 - e^{-0.32}) \\
&= 62,500(1 - 0.72615) \\
&= 17,115.63
\end{aligned}$$

■

Problem 9.12. A hive which can support a maximum of 7000 bees currently has 5000 growing exponentially at a constant rate of 5% a year . What will the population be in 4 years ?

Solution. Comparing with (9.11) we get $M = 7000, k = 5\% = 0.05$
Substituting in (9.11) we get,

$$\frac{dy}{dt} = 0.05(7000 - y)$$

Separate the variables and integrate,

$$\begin{aligned}
\int \frac{dy}{7000 - y} &= \int 0.05dt \\
-\ln(7000 - y) &= 0.05t + c \\
\ln(7000 - y) &= -0.05t - c \\
7000 - y &= e^{-0.05t - c} \\
7000 - y &= e^{-0.05t} \cdot e^{-c} \\
7000 - y &= Ae^{-0.05t} \quad (where\ A = e^{-c}) \\
-y &= Ae^{-0.05t} - 7000 \\
y &= 7000 - Ae^{-0.05t}
\end{aligned}$$

Thus we get,

$$y = 7000 - Ae^{-0.05t}. \tag{9.12}$$

Substituting $y(0) = 5000$ in (9.12) we get,

$$5000 = 7000 - Ae^{-0.05(0)} = 7000 - A \quad \Rightarrow A = 2000$$

Thus $y = 7000 - 2000e^{-0.05(4)} = 7000 - 2000(0.81873) \approx 5363.$ ■

9.10 Exercises

1. Use integration by substitution to evaluate the indefinite integral for the following functions:

   (a) $\int 3x^2(x^3+7)^5 dx$

   (b) $\int 128x(8x^2-9)^3 dx$

   (c) $\int \frac{7x^2x}{(7x^3-6)^3} dx$

   (d) $\int \frac{10x+3}{(5x^2+3x-7)^4} dx$

   (e) $\int 66x\sqrt[3]{8x^2-15} dx$

2. Use integration by substitution to evaluate the indefinite integral for the following functions:

   (a) $\int 80x^3 e^{5x^4} dx$

   (b) $\int x^4 e^{x^5} dx$

   (c) $\int (1-x^2)e^{3x-x^3} dx$

   (d) $\int \frac{(\ln x)^2}{x} dx$

   (e) $\int \frac{1}{x(\ln 7x)^3} dx$

3. Use integration by parts to evaluate the indefinite integral for the following functions:

   (a) $\int x(x-6)^4 dx$

   (b) $\int \frac{8x}{(x-5)^2} dx$

   (c) $\int \frac{6x}{\sqrt{x+7}} dx$

   (d) $\int \frac{7x}{e^{x+5}} dx$

   (e) $\int \frac{5x+2}{e^x} dx$

4. Use integration by parts to evaluate the indefinite integral for the following functions:

   (a) $\int \frac{\ln x}{x^3} dx$

   (b) $\int \ln x^5 dx$

   (c) $\int x(1-5x)^3 dx$

   (d) $\int 27x^2 e^{3x} dx$

5. Evaluate the improper integral $\int_{-\infty}^{-2} \frac{48}{x^4} dx$

6. Use L'hôpital's rule to evaluate the following limits:

   (a) $\lim_{x\to 3} \frac{\ln x}{e^{3x}}$

   (b) $\lim_{x\to\infty} \frac{2x^3-5}{x^2+4}$

   (c) $\lim_{x\to\infty} \frac{2x^2-3x}{5x^2-12}$

7. Calculate the following double integrals:

   (a) $\int_0^1 \int_0^1 6e^{2x+3y} dy dx$

   (b) $\int_{-3}^0 \int_{-1}^1 2xe^{xy} dy dx$

   (c) $\int_0^1 \int_0^{x^2} \frac{30y}{x^5+1} dy dx$

   (d) $\int_0^1 \int_{x^2}^3 8xe^2 y dy dx$

8. Calculate the following double integrals:

   (a) $\int_1^4 \int_0^y (18x+12y) dx dy$

   (b) $\int_0^3 \int_0^y \sqrt{xy} dx dy$

   (c) $\int_1^2 \int_y^{2y} \frac{1}{x} dx dy$

9. Approximate the value of the definite integral $\int_{-1}^{1} e^{3x} dx$ with the (1) Rectangle Rule, (2) Trapezoidal rule, (3) Simpson's rule for $n = 4$. Also check the accuracy of the answers by using integration.

10. Approximate the value of the definite integral $\int_{1.2}^{2.4} xe^{x} dx$ with the (1) Rectangle Rule, (2) Trapezoidal rule, (3) Simpson's rule for $n = 3$. Also check the accuracy of the answers by using integration.

11. Find (1) the general and (2) the particular solution for each of the following differential equations:

    (a) $y'(t) = 18e^{3t} - 8e^{2t} + 9,\ y(0) = -3$

    (b) $3\dfrac{dy}{dt} = 63\sqrt{t-12},\quad y(12) = 7$

    (c) $y'(t) = -3y,\ y(0) = 14$

    (d) $y'(t) = -0.25y,\ y(0) = 8$

12. Find the general solution for each of the following differential equations by separating the variables:

    (a) $\dfrac{dy}{dt} = t^4 y^2$

    (b) $y' = e^{t-y}$

    (c) $y' = 4t^3 y - 3t^2 y$

13. The rate at which the number of insects decreases after t hours following the introduction of a pesticide is given by

$$\frac{dy}{dt} = \frac{-540}{1+3t}$$

If there are 1000 insects initially, find their number at $t = 5$.

Text Books (As per Syllabus)

Edward T. Dowling: Calculus for Business, Economics and Social Sciences, Schaum's Outline Series, TMH, 2005.

Further Readings

1. Ismor Fischer: Basic Calculus Refresher, http://www.stat.wisc.edu/ ifischer/calculus.pdf

2. S.K Chung: Understanding Basic Calculus, http://www.mathdb.org/basic−calculus/BasicCalculus.pdf

3. Srinath Baruah : Basic Mathematics and its Applications in Economics, Macmillan.

4. Taro Yamane: Mathematics for Economists, Second ed., PHI.