FOURTH SEMESTER M.Sc DEGREE (MATHEMATICS) EXAMINATION,
JUNE 2012
(CUCSS-PG-2010)
MT4E02 : ALGEBRAIC NUMBER THEORY

MODEL QUESTION PAPER

Time: 3 hrs. Max. Weightage: 36

PART A

(Short Answer Type Questions)
Answer all the questions – Each question has weightage 1

1. Let R be a ring. Define an R-module.
2. Find the minimum polynomial of $i+\sqrt{2}$ over Q, the field of rationals.
3. Define the ring of integers of a number field K and give the one example.
4. Find an integral basis for $Q(\sqrt{5})$
5. Define a cyclotomic filed. Give one example
6. If $K = Q(\zeta)$ where $\zeta = e^{\frac{2\pi i}{5}}$, find $N_K(\zeta^2)$
7. What are the units in $Q(\sqrt{-3})$.
8. Prove that an associate of an irreducible is irreducible.
9. Define i) The ascending chain condition
   ii) The maximal condition
10. If x and y are associates, prove that $N(x) = \pm N(y)$
11. Define : A Euclidean Domain . Give an example.
12. Sketch the lattice in R^2 generated by (0,1) and (1,0)
13. Define the volume v(X) where $X \subset R^n$
14. State Kummer's Theorem.

(14 X 1 =14)

PART B

(Paragraph Type Questions)
Answer any seven questions-Each question has weightage 2

15. Express the polynomials $t_1^2 + t_2^2 + t_3^2$ and $t_1^3 + t_2^3$ in terms of elementary symmetric polynomials.

16. Prove that the set **A** of algebraic numbers is a subfield of the complex field **C.**

17. Find an integral basis and discriminent for $Q(\sqrt{d})$ if

    i) (d -1) is not a multiple of 4

    ii) (d -1) is a multiple of 4

18. Find the minimum polynomial of $\xi = e^{\frac{2\pi i}{p}}$, p is an odd prime , over Q and find its degree.

19. Prove that factorization into irreducibles is not unique in $Q(\sqrt{-26})$

20. Prove that every principal ideal domain is a unique factorization domain.

21. If **D** is the ring of integers of a number field **K**, and if **a** and b are non-zero ideals if **D**, then show that **N(ab)=N(a) N(b)**

22. State and prove Minkowski's theorem.

23. If $\alpha_1, \alpha_2, \alpha_3, ...\alpha_n$ is a basis for K over Q, then prove that $\sigma(\alpha_1), \sigma(\alpha_2), ...\sigma(\alpha_n)$ are linearly independent over R, where σ is a Q-algebra homomorphism.

24. Prove that the class group of a number filed is a finite abelian group and the class number h is finite.

(7 X 2 =14)

PART –C

(Essay Type Questions)
Answer any two questions-Each question has weightage 4

25. Prove that every subgroup H of a free Abelian group G of rank n is a free of rank $s \le n$. Also prove that there exists a basis $u_1, u_2, u_3, ...u_n$ for G and positive integers $\alpha_1, \alpha_2, \alpha_3, ...\alpha_s$ such that $\alpha_1 u_1, \alpha_2 u_2, \alpha_3 u_3, ...\alpha_s u_s$ is a basis for H.

26. a) If K is a number field, Then prove that $K = Q(\theta)$ for some algebraic number θ.

    b) Express $Q(\sqrt{2}, \sqrt{3})$ in the form of $Q(\theta)$

27. In a domain in which factorization into irreducible is possible prove that each factorization is unique if and only if every irreducible is prime.

28. Prove that an additive subgroup of R^n is a lattice if and only if it is discrete.

(2 X 4 = 8)