**FOURTH SEMESTER M.Sc DEGREE (MATHEMATICS) EXAMINATION, JUNE 2012**
**(CUCSS-PG-2010)**

## MT4E03: MEASURE AND INTEGRATION

**MODEL QUESTION PAPER**

Time: 3 hrs. Max. Weightage: 36

# Part A

(Short Answer Type Questions)
Answer all the questions. Each question carries 1 weightage.

1. Define Borel set.Verify whether the set N of natural numbers is a Borel set in R.
2. Let X be a measurable space. If $f = u + iv$ is a complex measurable function on X, show that f is a real measurable function on X.
3. If $A \subseteq B$ and $f \geq 0$ , prove that $\int_A f d\mu = \int_B f d\mu$ .
4. Show that if $\lambda$ and $\mu$ are complex measures and $\lambda \perp \mu$, then $|\lambda| \perp |\mu|$.
5. Give an example of a measure which is not complete.
6. Define inner- regular Borel set. Give an example.
7. Prove that every set of positive measure has non- measurable subsets.
8. Suppose that $\mu$ and $\lambda$ are measures on a $\sigma$- algebra M, $\mu$ is positive and $\lambda$ is complex. Prove that $\lambda << \mu$, if for every $\varepsilon > 0$ corresponds to a $\delta > 0$ such that $|\lambda(E)| < \varepsilon$ for all $E \in M$.
9. Define Lebesgue point of a function $f \in L^1(R^K)$ and give an example.
10. Let $x \in R^K$ and $\{E_i\}$ be a sequence of Borel sets in $R^K$. State the conditions under which $\{E_i\}$ shrinks nicely to x . Give an example.
11. Let $E \subseteq X \times Y$. Define the x - section $E_x$. If E is the unit circle, find $E_1$.
12. Define the product of two measures and state Fubini's theorem.
13. Let $f,g \in L^1(R^1)$. Show that the function defined by $F(x,y)=f(x-y)g(y)$ is a Borel function on $R^2$.
14. Let m be the Lebesgue measure on [a,b]. Define $\lambda(E) = i\,\mu(E)$. Then find the total variation $|\lambda|$ of $\lambda$.

(14 x 1 = 14)

# Part B

(Paragraph Type Questions)
Answer any seven questions. Each question carries 2 weightage

15. State and prove Fatous Lemma.
16. Define $L^1(\mu)$ and show that if $f \in L^1(\mu)$, then $| \int_X f d\mu | \leq |\int_Y f d\mu |$.

17. Suppose $\mu(X) < \infty$, $f \in L^1(\mu)$, S is a closed set in the complex plane and the average $A_E(f) = 1/\mu(E) \int f d\mu$ lies in S with $\mu(E) > 0$, then show that $f(x) \in S$ for almost all $x \in X$.
18. Show that Cc(X) is a vector space.
19. Let X be a locally compact Hausdorff space in which every open set is $\sigma$-compact. Let $\lambda$ be any positive Borel measure on X such that $\lambda(K) < \infty$ for every compact set K in X. Show that $\lambda$ is regular.
20. If $A \in R$ and every subset of A is Lebesgue measurable, prove that $m(A) = 0$.
21. Suppose $f \in L^1(\mu)$, f is real valued and $\varepsilon > 0$. Then prove that there exist functions u and v on X such that $u \leq f \leq v$, u is upper semi-continuous and bounded above, v is lower semi-continuous and bounded below and $\int_X (v-u) d\mu < \varepsilon$.
22. State and prove Hahn- Decomposition Theorem.
23. Suppose X is locally compact Hausdorff space, v is open in X, $K \subseteq V$ and K is compact. Then show that there is an $f \in Cc(X)$ such that $K \prec f \prec V$.
24. If $\mu$ is a complex measurable function on X show that $|\mu|(X) < \infty$.

(7 x 2 = 14)

## Part C

(Essay Type Questions)

Answer any two questions. Each question carries 4 weightage.

25. State and prove Lebesgue Monotone Convergence Theorem.
26. State and prove Lucins Theorem.
27. State and prove Lebesgue Radon- Decomposition Theorem.
28. Let $(X,s,\mu)$ and $(Y,\tau,\lambda)$ be $\sigma$- finite measure spaces. Suppose $Q \in s \times \tau$. If $\phi(x) = \lambda(Q_X)$, $\varphi(y) = \mu(Q^Y)$ for all $x \in X$ and $y \in Y$. Prove that $\phi$ is s- measurable and $\varphi$ is $\tau$- measurable and $\int_X \phi d\mu = \int_Y \varphi d\lambda$.

(2 x 4 = 8)