SECOND SEMESTER M.Sc.(MATHEMATICS) DEGREE EXAMINATION (CUCSS-PG-2010)

Model Question Paper

MT2C07-REAL ANALYSIS-II

Time 3hours Max.Weightage:36

PartA(Short Answer Type Questions)

Answer all questions

(Each Question has Weightage One)

1. Prove that to every $A \in L(\mathbb{R}^n, \mathbb{R}^1)$ corresponds a unique $y \in \mathbb{R}^n$ such that $Ax = x.y$, for every $x \in \mathbb{R}^n$. Prove that $\|A\| = |y|$

2. Find the derivative of a linear transformation $A : \mathbb{R}^n \to \mathbb{R}^m$ at each point of $\mathbb{R}^n$

3. Let $f : \mathbb{R}^3 \to \mathbb{R}^1$ be defined by $f(x, y, z) = x^2y+y^2z+z^2x$ for $(x, y, z) \in \mathbb{R}^3$ Find the gradient of f at $(1, 2, 3)$.

4. If $f : \mathbb{R}^2 \to \mathbb{R}^2$ is defined by $f(x, y) = (xy, x^2y^4), (x, y) \in \mathbb{R}^2$.Find $f'(0, 0)$

5. Does there exists an algebra which is not a σ-algebra.Justify your claim .

6. Suppose m is a countably additive measure on a σ-algebra $\mathscr{M}$ of subsets of a non-empty set X. If there is a set A in $\mathscr{M}$ with $m(A) < \infty$ then prove that $m(\emptyset) = 0$

7. Prove that the outer measure of a singleton set is zero.

8. Show that $\chi_{[0,1]}$ is measurable.

9. Prove or disprove : If f is a real function defined on [a,b], where $a < b$, such that f^2 is measurable, then f is measurable.

10. For n=1,2, . . . ,let

$$f_n(x) = \begin{cases} 0 & \text{if } x < n \\ 1 & \text{if } x \geq n \end{cases}$$

Verify Fatous lamma for the sequence (f_n)

11. Let f be a real measurable function defined on a measurable set E.Does the integrability of $|f|$ imply that of f.

12. If (f_n) is a sequence measurable functions defined on a measurable set E of finite measure and $f_n \to f$ a.e then prove that (f_n) converges in measure.

13. Let f be the function defined by $f(0) = 0$ and $f(x) = x\sin(\frac{1}{x})$ for $x \neq 0$.Find $D^+f(0)$.

14. Give an example of a continuous function which is not of bounded variation. Justify your answer?

Part B (Paragraph Type Questions)
Answer any Seven Questions.
(Each question has weightage two)

15. If S is a metric space if $a_{11}, a_{12} \ldots a_{mn}$ are real continuous function on S and if for each $p \in S, A_p$ is the linear transformation of $\mathbb{R}^n$ in to $\mathbb{R}^m$ whose matrix has entries $a_{ij}(p)$. Prove that the mapping $p \to A_p$ is a continuous mapping of S in to $L(\mathbb{R}^n, \mathbb{R}^m)$.

16. Let E be an open set in $\mathbb{R}^n$,f maps E into $\mathbb{R}^m$ and $x \in E$. When is f said to be differentiable at x and show that if f is differentiable at x then the derivative is unique.

17. Show by an example in the case of functions of several variables, existence of partial derivatives at a point does not imply the existence of the derivative at that point.

18. If X is a complete metric space ,and if φ is a contraction of X in to X then prove that there is one and only one $x \in X$ such that $\varphi(x) = x$.

19. Show that if f is a measurable real valued function and g a continues real function defined on $(-\infty, \infty)$ then gof is measurable.

20. Prove or disprove: If $\{f_n\}$ is a sequence of measurable functions defined on a measurable set E and suppose that there is a real number M such that $|f(x)| \leq M$ for all n and for all x and if $f(x) = \lim f_n(x)$ for each $x \in E$, then $\int_E f = \lim \int_E f_n$.

21. If f and g are two integrable functions over$E$, then prove that $f + g$ is integrable over E and $\int_E f + g = \int_E f + \int_E g$

22. Let (f_n) be a sequence of measurable functions that converges in measure to a function f.Prove that there is a subsequence (f_{n_k}) that converges to f almost everywhere.

23. Show that a function f is of bounded variation on $[a, b]$ iff f is the difference of two monotone real-valued functions on $[a, b]$.

24. Show that if f is absolutely continuous then f has a derivative almost everywhere.

PartC (Essay Type Questions)
Answer Any Two Questions
(Each Question has Weightage Four)

25. (a) Suppose f is a $\mathscr{C}'$ mapping of an open set $E \subset \mathbb{R}^n$ into $\mathbb{R}^n$, $f'(a)$ is invertible for some $a \in E$ and $b = f(a)$ .Then prove that there exists open sets U and V in $\mathbb{R}^n$ such that $a \in U, b \in V, f$ is one-to-one on U and $f(U) = V$.

    (b) Show that the continuity of f' is needed in the inverse function theorem.

    (c) Show that the invertibility of f' at every point of E in (a) need not imply that f is one to one on E.

26. (a) When is a subset E of $\mathbb{R}$ said to be (Lebesgue) measurable. Illustrate if with an example.

    (b) Construct a non measurable subset of $\mathbb{R}$.

    (c) Show that if A is any set with $m^*A > 0$ then there is a non measurable set $E \subset A$.

    (d) Does there exists any sequence (E_i) of sets in $\mathbb{R}$ with $E_i \supset E_i + 1$, $m^*(E_i) < \infty$ and $m * (\cap E_i) < \lim m^*(E_i)$.

27. (a) State and prove a necessary and sufficient condition for a function f defined bounded on a measurable set E with $m(E)$ finite to be measurable.

    (b) Show that every Riemann integrable function f is measurable and Lebgesgue integrable. Also show that for such function f

$$R \int_a^b f(x)dx = \int_a^b f(x)dx$$

28. (a) Let E be a set of finite outer measure and $\mathcal{I}$ a collection of intervals that cover E in the sense of Vitali. Then prove that for every $\epsilon > 0$ there is a finite disjoint collection $\{I_1, I_2, \ldots, I_N\}$ of intervals in $\mathcal{I}$ such that $m^*(E \setminus \cup_{n=1}^{N} I_n) < \epsilon$

(b) Using Vitali's Lemma prove that if f is absolutely continues and $f'(x) = 0$ a.e then f is constant.

SECOND SEMESTER M.Sc(MATHEMATICS) DEGREE EXAMINATION, CUCSS-PG-2010

Model Question Paper

MT2C06-ALGEBRA-II

Time 3hours **Max.Weightage:36**

PartA(Short Answer Type Questions)

Answer ***all*** questions

(Each Question has Weightage ***One***)

1. Which of the following rings are fields? Justify your claim.
   a) $\mathbb{Q}[x]/ < x^{100} - 1 >$ b) $\mathbb{Q}[x]/ < x^{100} - 2 >$.
2. Find $irr(\sqrt{\frac{1}{3} + \sqrt{7}}, \mathbb{Q})$.
3. Prove that there is no proper subfield of $\mathbb{Q}(\sqrt[3]{2})$ properly containing $\mathbb{Q}$
4. Prove or disprove: $\mathbb{Q}(\sqrt[4]{3})$ is a field of constructible real numbers.
5. Find all irreducible polynomials of degree 2 in $Z_2[x]$.
6. Find all extensions of the automorphism $\Psi_{\sqrt{3},-\sqrt{3}}$ of $\mathbb{Q}(\sqrt{3})$ to isomorphisms of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, i)$ onto subfields of $\mathbb{Q}$.
7. Give an example for an extension E of $\mathbb{Q}(\sqrt[4]{2})$ such that $[\mathbb{E} : \mathbb{Q}] = \{\mathbb{E} : \mathbb{Q}\} =\mid G(\mathbb{E}/\mathbb{Q}) \mid$. Justify your claim.
8. Describe an automorphism σ of $\mathbb{Q}(\pi)$ which maps π onto $-\pi$. Also find its fixed field.
9. Prove that every finite field is a normal extension of some $\mathbb{Z}_p$.
10. Prove or disprove: Every algebraically closed field is perfect.
11. Prove that $G(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})/\mathbb{Q}(i\sqrt{3}))$ is a normal subgroup of $G(\mathbb{Q}(\sqrt[3]{2}, i\sqrt{3})/\mathbb{Q})$
12. Find $\Phi_8(x)$ over $\mathbb{Q}$.

13. Is the regular 150-gon constructible? Why?

14. Let $\mathbb{K}$ be the splitting field of $x^2 + x + 1$ over $\mathbb{Z}_2$. Is $\mathbb{K}$ an extension of $\mathbb{Z}_2$ by radicals? Why?

PartB(Paragraph Type Questions)

Answer ***any seven*** questions

(Each Question has Weightage ***two***)

15. Prove that any ring, with unity of characteristic zero contains a subring isomorphic to $\mathbb{Z}$. Find a subring of $M_n(\mathbb{R})$ which is isomorphic to $\mathbb{Z}$.

16. Construct a field F of 8 elements. What is $G(F/\mathbb{Z}_2)$.

17. Find a basis for the splitting field K of $x^4 - 2$ over $\mathbb{Q}$.

18. Prove that an algebraic extension E of F is finite if and only if there exist a finite number of elements $\alpha_1, \alpha_2, ..., \alpha_n$ in E such that $E = F(\alpha_1, \alpha_2, ..., \alpha_n)$.

19. Prove that any finite field has exactly p^n elements for some prime p and a positive integer n.

20. Give an example of a finite field extension $F \leq E$, where E is not a separable extension of F.

21. Prove that if E is a finite extension of F, then the number of extensions of an isomorphism of F to an isomorphism of E is completely determined by F and E.

22. Let $F \leq E \leq K$, where K is a finite normal extension of F. Then prove or disprove:

    (a) K is a normal extension of E.

    (b) E is a normal extension of F.

23. Describe the Galois group of the n^{th} cyclotomic extension of $\mathbb{Q}$ over $\mathbb{Q}$.

24. Prove the insolvability of the quantic.

PartC(Essay Type Questions)

Answer ***any two*** questions

(Each Question has Weightage ***four***)

25. (a) Prove that an ideal $< p(x) > \neq \{0\}$ of $F[x]$ is maximal if and only if $p(x)$ is irreducible over F.

    (b) Describe the maximal ideals of $\mathbb{C}[x]$.

    (c) Prove the impossibility of squaring a circle.

26. (a) State and prove the conjugation isomorphism theorem.

    (b) Prove that if F is a finite field of characteristic p and σ_p is the Frobenius automorphism of F defined by $\sigma_p(a) = a^p$, for $a \in F$, then the fixed field of σ_p in F is isomorphic to $\mathbb{Z}_p$.

27. (a) Prove that every finite separable extension of a field F is a simple extension of F.

    (b) Prove that every irreducible polynomials in $Z_p[x]$ divides $x^n - 1$ for some n.

28. (a) Describe the group of the polynomial $x^3 - 2$ over $\mathbb{Q}$.

    (b) Prove that for every prime p, $x^p - 1$ is solvable by radicals over $\mathbb{Q}$.

SECOND SEMESTER M.Sc. (MATHEMATICS) DEGREE EXAMINATION (CUCSS PG-2010)

Model Question Paper

MT 2C10 Number Theory

Time: 3 Hours | Maximum Weightage: 36

Part A (Short Answer Type Questions)

Answer All Questions

Each question has weightage one

1. Find all integers n such that $\phi(n) = \frac{n}{2}$.
2. Let f be a multiplicative function. Show that $f^{-1}(p) = -f(p)$ if p is prime.
3. Prove that $[2x] - 2[x]$ is either 0 or 1.
4. Prove that for every $n > 1$, there exist n consecutive composite numbers.
5. Show that an integer $n > 0$ is divisible by 3 if and only if sum of its digits is divisible by 3.
6. State Wolstenholme's theorem and verify it for $p = 7$.
7. Determine the quadratic residues and non residues modulo 7.
8. Determine whether 73 is a quadratic residue or nonresidue of the prime 383.
9. Find a formula for the number of different affine enciphering transformations on single letter message units in an $N-$ letter alphabet.
10. Find the inverse of the matrix $\begin{pmatrix} 1 & 3 \\ 4 & 3 \end{pmatrix}$ mod 5.
11. How do we send a signature in RSA cryptosystem.
12. What is the knapsack problem ? When it is said to be superincreasing?
13. For every integer $n \geq 1$, prove that $log\ n = \sum_{d|n} \Lambda(d)$.
14. Prove that the Dirichlet product of two multiplicative functions is multiplicative.

Part B (Paragraph Type Questions)

Answer any seven Questions

Each question has weightage two

15. Calculate the highest power of 10 that divides 1000!.

16. State and prove Little Fermat theorem.

17. Find all x which simultaneously satisfy the system of congruences $x \equiv 1(\mod 3), x \equiv 2(\mod 4), x \equiv 3(\mod 5)$.

18. Define the Legendre symbol $(n|p)$ and show that it is a completely multiplicative function of n.

19. State (without proof) Gauss Lemma and use it to compute $(5|17)$.

20. Let P be an odd prime, prove that $\sum_{r=1(r|p)}^{p-1} r = \frac{p(p-1)}{4}$, if $p \equiv 1(\mod 4)$.

21. P is an odd positive integer, show that $(2|P) = (-1)^{(p^2-1)/8}$.

22. In the 27-letter aplhabet(with blank=26) use the affine enciphering transformation with key $a = 13, b = 9$ to encipher the message" HELP ME".

23. Find all solutions $\begin{pmatrix} x \\ y \end{pmatrix}$ mod 9, writing x and y as non negative integers less than 9 of the system

$$x + 4y \equiv 3(\mod 9)$$
$$5x + 8y \equiv 2(\mod 9)$$

24. Write a note on the ElGamal cryptosystem.

Part A (Essay Type Questions)

Answer any two Questions

Each question has weightage four

25. (a) State and prove Mobius inversion formula.
    (b) Let f be a multiplicative. Prove that f is completely multiplicative if and only if $f^{-1}(n) = \mu(n)f(n)$ for all $n \geq 1$.

26. (a) State and prove Abel's identity.
    (b) For every integer $n \geq 2$, prove that $\frac{1}{6}\frac{n}{logn} < \pi(n)$.

27. (a) Assuming the Gauss's lemma and the formula for the integer m in the Gauss's lemma, prove quadratic reciprocity law.

(b) Determine those odd primes p for which 3 is a quadratic residue and those for which it is a nonresidue.

28. (a) Describe the Silver-Pohlig- Hellman Algorithm for computing discrete logarithms in finite fields.

(b) Using the algorithm in part(a), find the discrete log of 28 to the base 2 in F_{37}^* (2 is a generator of F_{37}^*)

SECOND SEMESTER M.Sc.(MATHEMATICS) DEGREE EXAMINATION, CUCSS-PG-2010

Model Question Paper

MT2C09 PDE and Integral Equations

Time: 3 Hours Maximum. Weightage: 36

Part A (Short Answer Type Questions)

Answer *all* questions

Each question has weightage *one*

1. Differentiate between *linear* and *non linear* partial differential equations. Give one example for each.
2. Differentiate between *quasi linear* and *semi linear* partial differential equations. What is the general form of a quasi linear equation in two independent variables.
3. What is the difference between the complete integral and the general integral of a PDE.
4. What is Cauchy Problem. Give an example of a Cauchy Problem ?
5. Define a Pfaffian Differential equation. Give an example of a Pfaffian differential equation in two variables. When does a Pfaffian Differential equation integrable?
6. State the Neumann problem and show that the solution of the Neumann problem is unique up to the addition of a constant.
7. Verify that $u(x, y) = n^{-2} sinhny \sin nx$ is the solution of the equation $u_{xx} + u_{yy} = 0; u(x, 0) = 0, u_y(x, 0) = n^{-1} \sin nx$. Prove that this solution is not stable.
8. State Dirichlet problem and show that Dirichlet problem is stable.
9. State the Neumann Problem for the Upper half plane.
10. State Harnack's Theorem.
11. Differentiate between Fredholm and Voltera Integral Equations. Give one example for each.
12. Differentiate between Resolvant Kernel and Separable Kernels.
13. Convert the differential equation $y'' + y = 0; y(0) = y'(0) = 0$ into an integral equation.
14. If $y'' = F(x)$, and y satisfies the end conditions $y(0) = 0$ and $y(1) = 0$, show that

$$y(x) = \int_0^x (x - \xi)F(\xi)d\xi - x \int_0^1 (1 - \xi)F(\xi)d\xi.$$

Part B (Paragraph Type Questions)
Answer any *seven* questions
Each question has weightage *two*

15. Eliminate the arbitrary function F from the equation $F(x+y, x-\sqrt{z}) = 0$ and find the corresponding PDE.

16. Prove that the Pfaffian differential equation
$$(y^2 = yz)dx + (xz + z^2)dy + (y^2 - xy)dz = 0$$
is integrable.

17. Find the integral surface of the equation $(2xy-1)p+(z-2x^2)q = 2(x-yz)$ which passes through the line $x_0(s) = 1, y_0(s) = 0$ and $z_0(s) = s$.

18. Define the Monge cone at a point (x_0, y_0, z_0) characterized by the differential equation $f(x, y, z, p, q) = 0$ and find the Monge cone at $(0, 0, 0)$ for the differential equation $p^2 + q^2 = 1$.

19. Reduce the equation $x^2u_{xx} - y^2u_{yy} = 0$ into its canonical form and solve it.

20. Derive the D'Alembert's solution which describes the vibration of an infinite string.

21. Show that the solution for the Dirichlet problem for a circle of radius a is given by the Poisson integral formula.

22. Show that the characteristic numbers of the Fredholm integral equation with a real symmetric Kernel are real.

23. Solve the Fredhom integral equation by iterative method:
$$y(x) = 1 + \lambda \int_0^1 (1 - 3x\xi)y(\xi)d\xi$$

24. Transform the problem $y'' + xy = 1, y(0) = y(1) = 0$ to the integral equation $y(x) = \int_0^1 G(x,\xi)\xi y(\xi)d\xi - \frac{1}{2}x(1-x)$ where
$$G(x,\xi) = \begin{cases} x(1-\xi) & \text{when} \quad x < \xi \\ \xi(1-x) & \text{when} \quad x > \xi \end{cases}$$

Part C (Essay Type Questions)
Answer any *two* questions
Each question has weightage *four*

25. (a) Show that the equations

$$f = xp - yq - x = 0$$
$$g = x^2p + q - xz = 0$$

are compatible and find a one parameter family of common solutions.

(b) Find a complete integral of the equation $(p^2 + q^2)x = pz$ and the integral surface containing the curve C: $x_0 = 0, y_0 = s^2, z_0 = 2s$.

26. (a) Find the characteristic strips of the equation $xp + yq - pq = 0$.

(b) Suppose that $u(x, y)$ is harmonic in a bounded domain D and continuous in $\overline{D} = D \cup B$ where B is the boundary of D. Show that u attains its maximum and minimum on the boundary B.

27. (a) Solve the following problem:

$$\nabla^2 u = u_{xx} + u_{yy} = 0, 0 \le x \le a, 0 \le y \le b$$

with boundary conditions $u(x, 0) = f(x), u(x, b) = u(0, y) = u(a, y) = 0$

(b) Find the solution of the equation

$$z = \frac{1}{2}(p^2 + q^2) + (p - x)(q - y)$$

which passes through the x- axis

28. (a) Show that Green's function for $y'' = 0$, $y(0)+y(1) = 0, y'(0)+y'(1) = 0$ is

$$G(x, \xi) = \begin{cases} (1 - \xi) & \text{when} \quad 0 < x < \xi \\ (1 - x) & \text{when} \quad \xi < x \le 1 \end{cases}$$

(b) Show that the Integral equation

$$y(x) = f(x) + \frac{1}{\pi} \int_0^{2\pi} \sin(x + \xi) y(\xi) d\xi$$

possess no solution for $f(x) = x$, but that it possesses infinitely many solutions when $f(x) = 1$.

SECOND SEMESTER M.Sc(MATHEMATICS) DEGREE EXAMINATION, CUCSS-PG-2010

Model Question Paper

MT2C08-TOPOLOGY-I

Time 3hours **Max.Weightage:36**

PartA(Short Answer Type Questions)

Answer ***all*** questions

(Each Question has Weightage ***One***)

1. Prove or disprove that the indiscrete space is not obtainable from a metric.
2. Prove that a space is second countable if and only if it has a countable sub-base.
3. Show that the finite product of second countable space is second countable.
4. Find the topology induced by the discrete metric.
5. If A is homeomorphic to a subspace of B and B is homeomorphic to a subspace of A, can you conclude that A is homeomorphic to B. Justify.
6. Show that the T_1 axiom is equivalent to the requirement that finite point sets be closed.
7. Show that ΠX_α is connected and nonempty, then each X_α is connected.
8. Prove or disprove that in a metric space, a closed ball is the closure of the open ball with the same centre and radius.
9. Characterise clopen sets in terms of boundaries.
10. Prove or disprove that the interior and the boundary of a connected set are connected.

11. Prove that normality is a weakly hereditary property.

12. Prove that the unit circle S^1 is compact.

13. Prove that the co-countable topology on a set makes it into a Lindeloff space.

14. Prove or disprove that $\mathbb{R}$ and $[a, b], a < b$ are homeomorphic.

PartB(Paragraph Type Questions)

Answer ***any seven*** questions

(Each Question has Weightage ***two***)

15. Show that metrisability is a hereditary property.

16. Define closure of a set in a topological space.
Show that a subset of a topological space is open iff it is a neighbourhood of each of its points.

17. Prove that every closed, surjective map is quotient map.

18. Prove that every continuous real valued function on a compact spaces is bounded and attains its extrema. What about the converse? Justify.

19. Show that every second countable space is first countable. What about the converse? Justify.

20. Prove that the product of two connected topological spaces is connected.

21. Show that every path-connected space is connected

22. Show that regularity is a hereditary property.

23. Define component of a topological space and give an example. Prove that compononts are closed sets and every nonempty connected subset is contained in a unique component.

24. Suppose a topological space X has the property that for every closed subset A of X, every continuous real valued function on A has a continuous extension to X. Then prove that X is normal.

PartC(Essay Type Questions)

Answer ***any two*** questions

(Each Question has Weightage ***four***)

25. (a) Show that a subset of $\mathbb{R}$ is disconnected iff it is not an interval.

    (b) Prove that every quotient space of a locally connected space is locally connected.

26. (a) Show that in a Hausdorff space, limits of sequences are unique.

    (b) Show that all metric spaces are T_4.

27. (a) Show that every regular, Lindeloff space is normal.

    (b) Show that a continuous bijection from a compact space onto a Hausdorff space is homeomorphism.

28. (a) State the Urysohn Characterisation of Normality.

    (b) Let X be a completely regular space. Suppose F is a compact subset of X, C a closed subset of X and $F \cap C = \emptyset$. Then prove that there exists a continuous function from X into the unit interval which takes the value 0 at all points of F and the value 1 at all points of C.