THIRD SEMESTER M. Sc. DEGREE (MATHEMATICS) EXAMINATION
(CUSS PG 2010)
MODEL QUESTION PAPER
MT3C11: COMPLEX ANALYSIS

TIME:3 HOURS Maximum weightage:36

PART A (Short Answer Type Question **1-14**)
Answer All Questions. Each Question has Weightage **1**)

1. Show that the composition of two linear transformation is a linear transformation.
2. Find the fixed points of the transformation $w = \frac{z}{2-z}$.
3. Find the image of the line $y = y_0$ under the mapping $w = z^2$.
4. Use complex integration to find the length of a circle with centre 'a' and radius 'ρ'.
5. Compute $\int\limits_{|z|=1} \frac{e^z}{z} dz$.
6. Determine the nature of the singularity of $e^{1/z}$ at $z = 0$.
7. If $f(z)$ and $g(z)$ have algebraic orders h and k at $z = a$, show that fg has the order $h + k$.
8. Find the residue of $\frac{e^z}{(z-a)^2}$ at $z = a$.
9. How many roots does the equation $z^7 - 2z^5 + 6z^3 - z + 1 = 0$ have in the disk $|z| < 1$.
10. Find the Cauchy principal value of the integral $\int\limits_{-\infty}^{\infty} \frac{e^{ix}}{x} dx$.
11. If u is harmonic in a region Ω, show that $\int\limits_{\gamma} {}^*du = 0$ for all cycle γ which are homotopic to zero in Ω.
12. Find the Taylor series of $log(1 + z)$ about the origin, choosing the branch which is equal to zero at the origin.
13. Show that an elliptic function without poles is a constant.
14. Define Weierstrass sigma function $\sigma(z)$ and show that it is an odd function.

PART B (Paragraph Type Questions **15-24**)
Answer any Seven Questions. Each Question has Weightage **2**)

15. Prove that an analytic function in a region Ω, whose derivative vanishes identically must reduce to a constant.

16. State and prove the Symmetry principle.

17. Define the complex line integral of $f(z)$ over a piecewise differentiable curve γ and show that this integral is invariant under a change of parameter.

18. State and prove Morera's theorem.

19. Suppose $f(z)$ is analytic in the region Ω' obtained by omitting a point a from the region Ω. Prove that a is a removable singularity of $f(z)$ if and only if $\lim\limits_{z \to a} (z-a)f(z) = 0$.

20. Evaluate $\int\limits_0^{\pi} \frac{d\theta}{a+cos\theta}$, $a > 1$, by the method of residues.

21. Prove that the arithmetic mean of a harmonic function over concentric circles $|z| = r$ is a linear function of $logr$.

22. Show that the Laurent development of $\frac{1}{e^z-1}$ at the origin is of the form $\frac{1}{z} - \frac{1}{2} + \sum\limits_{k=1}^{\infty} (-1)^{k-1} \frac{B_k}{(2k)!} z^{2k-1}$, where B_k are the Bernoulli numbers.

23. Show that the sum of residues of an elliptic function is zero.

24. Prove that the addition theorem for the $\mathcal{P}$ function $\mathcal{P}(z+u) = -\mathcal{P}(z) - \mathcal{P}(u) + \frac{1}{4}\left(\frac{\mathcal{P}'(z)-\mathcal{P}'(u)}{\mathcal{P}(z)-\mathcal{P}(u)}\right)^2$.

PART C (Essay Type Questions **25-28**)

Answer any two Questions. Each Question has Weightage **4**)

25. (a) Define cross ratio and show that it is invariant under a linear transformation.

    (b) Prove that the cross ratio (z_1, z_2, z_3, z_4) is real if and only if the four points lie on a circle or on a straight line. Deduce that a linear transformation carries circles into circles.

26. (a) State and prove cauchy's theorem for a rectangle.

    (b) Compute $\int\limits_{|z|=1} |z-1||dz|$.

27. (a) State and prove the Residue theorem.

    (b) Evaluate $\int\limits_0^{\infty} \frac{x^2}{x^4+5x^2+6} dx$ by the method of residues.

28. (a) Prove that a discrete module consists of either zero alone, or of the integral multiples of a single complex number $w \neq 0$, or of all linear combination $n_1 w_1 + n_2 w_2$ with integral coefficients of two numbers w_1 and w_2 with nonreal ratio w_2/w_1.

    (b) With usual notation derive the differential equation $(\mathcal{P}'(z))^2 = 4(\mathcal{P}(z))^3 - g_2\mathcal{P}(z) - g_3$.

THIRD SEMESTER M.Sc DEGREE (MATHEMATICS) EXAMINATION

(CUCSS-PG-2010)

MODEL QUESTION PAPER

MT3C12: FUNCTIONAL ANALYSIS - I

Time: 3 hrs. Max. Weightage: 36

PART - A

Answer all questions from Part A. Each question carries a weightage 1

(1) Prove or disprove: The set $\mathbb{N}$ of natural numbers considered as a subset of the real line with the discrete metric is bounded.

(2) Let d and d' be two metrics on a non empty set X. If d is stronger than d' then prove that every open subset of X with respect d' is also open with respect to d.

(3) Show that there are divergent sequences (x_n) in l^p such that the sequence $(x_n(j))$ converges in the scalar field for each $j = 1, 2, \ldots$.

(4) Prove or disprove: All essentially bounded measurable functions are bounded.

(5) If $X = c_{oo}$ with the norm $\|.\|_\infty$ and $E = \{x \in X : |x(j)| \leq \frac{1}{j}, j = 1, 2, \ldots\}$ prove that E spans X but $E^o = \varnothing$.

(6) Show that a linear map from a normed space X into a normed space may be closed without it being continuous.

(7) Show that the map $f : X \to \mathbb{K}$ defined by $f(x) = \sum_{j=1}^{\infty} x(j)$ for $x = (x(1), x(2), \ldots)) \in X$ is discontinuous, where X is with the norm $\|.\|_2$.

(8) Prove that among all the l^p, $1 \leq p \leq \infty$ spaces, only l^2 an inner product space.

(9) Let X be an inner product space and $\{x_1, x_2\}$ an orthogonal set in X. Prove that $\|x_1 + x_2\|^2 = \|x_1\|^2 + \|x_2\|^2$.

(10) Let$< .,. >$ be an inner product on a linear space X. Let $x \in X$ Prove that $< x, y >= 0$ for all $y \in X$ if and only if $x = 0$.

(11) If X is a nonzero normed linear space then prove that X' is nonzero.

(12) Let τ be a metric space, $B(\tau)$ the linear space of all $\mathbb{K}$- valued functions on τ with sup norm and $C(\tau)$, the subspace of $B(\tau)$ consisting of all bounded continuous functions. Show that every Hann Banach extension of any positive linear functional on $C(\tau)$ to $B(\tau)$ is again positive.

(13) Let X be a normed space with its dual X' . Suppose X'' denotes the second dual of X. Then prove that the function $J : X \to X''$ defined by $J(x)(x'') = x'(x)$ for $x \in X, x' \in X'$ is an isometry.

(14) What is meant by completion of a normed space X. Illustrate it with an example.

PART - B

Answer any 7 questions. Each question has a weightage 2.

(15) Define a complete metric space and prove that the property of completeness of a metric space may not be shared by an equivalent metric.

(16) State Korovkin's theorem and deduce that the set of polynomials is dense in $C[a, b]$ with the sup metric.

(17) Let $x \in L^1([-\pi, \pi])$. Show that $\hat{x}(n) \to 0$ as $n \to \infty$

(18) Let X_1 be a closed subspace and X_2 be a finite dimensional subspace of a normed space X. Then prove that $X_1 + X_2$ is closed in X.

(19) Let X and Y be normed spaces and $F : X \to Y$ be a linear map. Show that F is continuous if the zero space $Z(F)$ is closed in X and the linear map $\tilde{F} : X/Z(F) \to Y$ defined by $\tilde{F}(x + Z(F)) = F(x)$ is continuous.

(20) Let $H = L^2([0, 1])$ be the Hilbert space with the inner product $< x, y >= \int_0^1 x(t)\overline{y(t)}dt$ for $x, y \in L^2([0, 1]$. Show that $\{1, \sqrt{2}\cos \pi t, \sqrt{2}\cos 2\pi t, \ldots\}$ forms an orthonormal basis for H.

(21) Let X be an inner product space, F a subspace of X and $x \in X$. Then prove that $y \in F$ is a best approximation from F to x if and only if $x - y \perp F$.

(22) Let $X = \mathbb{K}^2$ with the norm $\|.\|_\infty$ and $Y = \{(x(1), x(2)) : x(1) = x(2)\}$. Find all Hann Banach extensions of the continuous linear functional g on Y defined by $g(x(1), x(2)) = x(1)$.

(23) Prove that there does not exists any denumerable basis for a Banach space.

(24) Let X be a normed space and E be a subset of X. Them prove that E is bounded in X if and only if $f(E)$ is bounded in $\mathbb{K}$ for every $f \in X'$.

PART - C

Answer any two questions. Each question has weightage 4.

(25) (a) Let x be a continuous $\mathbb{K}$- valued function on $[-\pi, \pi]$ such that $x(\pi) = x(-\pi)$. Prove that the sequence of arithmetic means of the partial sums of the Fourier series of x converges uniformly on $[-\pi, \pi]$.

(b) Let E be a measurable subset of $\mathbb{R}$ with finite measure. If $1 \leq p < r < \infty$ then prove that $L^r(E) \subset L^p(E)$ and the inclusion function from $L^r(E) \to L^p(E)$ is continuous.

(c) Show by an example that this is not the case when $m(E) = \infty$

(26) Let $\{u_\alpha\}$ be an orthonormal set in a Hilbert space H. Prove that the following conditions are equivalent.

(i) $\{u_\alpha\}$is an orthonormal basis for H,

(ii) For every $x \in H$, the set $\{u_\alpha : < x, u_\alpha > \neq 0\}$ is countable and if $\{u_\alpha : < x, u_\alpha > \neq 0\} = \{u_1, u_2, ...\}$, then $x = \sum_n < x, u_n > u_n$,

(iii) For every $x \in H$ we have $\|x\|^2 = \sum_n | < x, u_n > |^2$, where $\{u_1, u_2, ...\} = \{u_\alpha : < x, u_\alpha > \neq 0\}$,

(iv) Span u_α is dense in H,

(v) If $x \in H$ and $< x, u_\alpha >= 0$ for all α then $x = 0$.

(27) Let X be normed space over $\mathbb{K}$, E be a non empty open convex subset of X and Y be a subspace of X such that $E \cap Y = \emptyset$. Prove that there exists an $f \in X'$ such that $f(x) = 0$ for every $x \in Y$ but $Re\ f(x) \neq 0$ for every $x \in Y$ and deduce that if E_1 and E_2 are two non empty disjoint convex subsets of

X, where E_1 is open in X, then there exists an $f \in X'$ and $c \in \mathbb{R}$ such that $Re\ f(x_1) < c < Re\ f(x_2)$ for all $x_1 \in E_1$ and $x_2 \in E_2$

(28) (a) Let X be a Banach space, Y be a normed space and $\mathscr{F}$ be a subset of $BL(X,Y)$ such that for each $x \in X$, the set $\{F(x) : F \in \mathscr{F}\}$ is bounded in Y. Prove that $\sup\{\|F\|, F \in \mathscr{F}\} < \infty$ and deduce that if (F_n) is a sequence in $BL(X,Y)$ such that the sequence $(F_n(x))$ converges in Y for every $x \in X$ to say, $F(x)$ then prove that $\|F\| \leq \liminf_{n\to\infty} \leq \sup\{\|F_n\|, n = 1, 2, \ldots\} < \infty$

(b) Show by examples both the results in (a) need not work if X not a Banach space.

THIRD SEMESTER M.Sc (MATHEMATICS) DEGREE EXAMINATION(CUCSS-PG-2010)

MODEL QUESTION PAPER
MT3C13: TOPOLOGY – II

Time : 3 hrs. **Max. Weightage: 36**

PART – A (Short Answer Type Question)
Answer *all* Questions
(Each questions has weightage **One**)

1. Prove that if the product is non-empty then each projection function is onto.

2. Prove or disprove : The product of open sets is open in the product topology.

3. Give an example of a bounded metric on the real line R which induces the usual topology on R. Substantiate your claim.

4. Prove or disprove: Let $\{f_i:X\to Y_i / i\in I\}$ be a family of functions and $e:X\to\Pi_{i\in I} Y_i$. If e is open, then $\{f_i:X\to Y_i / i\in I\}$ distinguishes points from closed sets in X.

5. Give an example of a space which is second countable and T_2, but not metrisable.

6. Show that if $h,h^1: X\to Y$ are homotopic and $k,k^1: Y\to Z$ are homotopic, then $k\circ h$ and $k^1\circ h^1$ are homotopic.

7. Check whether the map $p:R_+\to S^1$ given by $p(x)=(\cos 2\pi x,\sin 2\pi x)$ is a covering map.

8. Prove that if X is any convex subset of R^n, then $\pi_1(X,x_0)$ is the trivial group for any $x_0\in X$.

9. Prove or disprove: local compactness is a hereditary property.

10. Prove that one point compactification of the Euclidean space R^n is homeomorphic to S^n.

11. Prove or disprove: countable union of nowhere dense subset of a topological space is no where dense.

12. Prove that a non empty complete metric space is of second category.

13. Show by an example that total boundedness is not a topologically invariant.

14. Prove or disprove: completeness of a metric is a topological property.

PART – B (Paragraph Type Questions)
Answer any *seven* questions
(Each question has Weightage **two**)

15. Show that a locally connected space is the topological sum of the family of its components .

16. Prove that if the product is non-empty,then each coordinate space is embeddable in it.

17. Prove that the evaluation map is one – one if and only if the family $\{f_i : i \in I\}$ of maps distinguishes points.

18. Prove that a topological space is Tychonoff if and only if it is embeddable into a cube.

19. Let X be a compact, Hausdorff, totally disconnected space. Prove that X has a base consisting of clopen sets.

20. Let x_0 and x_1 be two given points of the path connected space X. Show that $\pi_1(X,x_0)$ is abelian if and only if for every pair α and β of paths from x_0 to x_1 satisfy $\hat{\alpha} = \hat{\beta}$, where $\hat{\alpha}$ and $\hat{\beta}$ are the functions from $\pi_1(X,x_0)$ to $\pi_1(X,x_1)$ by the equation $\hat{\alpha}([f]) = [\overline{\alpha}] * [f] * [\alpha]$ and $\hat{\beta}([f]) = [\overline{\beta}] * [f] * [\beta]$

21. Show that if A is a strong deformation retract of X, and B is a strong deformation retract of A then B is a strong deformation retract of X.

22. Prove that every compact metric space is complete. What about the converse?

23. Show that every complete metric space is the completion of any of its dense metric subspaces.

24. Prove that a subset of Euclidian space is compact if and only if it is closed and bounded.

PART C (Essay Type Questions)

Answer any *two* questions

(Each Question has Weightage **four**)

25. Let (X,T) be a normal space, A be a closed subset of X and f:A→R be a continuous function.

    a) Prove that f can be continuously extended to the whole space.

    b) What happens when A is not closed?

26. a) Show that a product of spaces is connected if and only if each co-ordinate space is connected.

    b) Prove or disprove: Local connectedness is a productive property.

27. Prove that the fundamental group of the circle is infinite cyclic.

28. a) Prove that sequential compactness is a countably productive property.

    b) Show that sequential compactness is preserved under continuous functions.